MOTIVIC INTEGRATION OVER DELIGNE-MUMFORD STACKS

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Abstract. The aim of this article is to develop the theory of motivic integration over Deligne-Mumford stacks and to apply it to the birational geometry of Deligne-Mumford stacks.

Contents

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1. Introduction

In this article, we study the motivic integration over Deligne-Mumford stacks, which was started in [Yas1]. The motivic integration was introduced by Kontsevich [Kon] and developed by Denef and Loeser [DL1], [DL2] etc. It is now well-known that the motivic integration is effective in the study of birational geometry. For example, Batyrev [Bat] has applied it to the study of stringy E-functions and Mustaţă [Mus] to one of the singularities appearing in the minimal model program.

We first recall the motivic integration over varieties. Thanks to Sebag [Seb], we can work over an arbitrary perfect field k. Let X be a variety over k, that is, a separated algebraic space of finite type over k. For a non-negative integer n, an *n-jet* of X over a k-algebra R is a $R[[t]]/t^{n+1}$-point of X. For each n, there exists an algebraic space J_nX parameterizing n-jets. For example, J_0X is X itself and J_1X is the tangent bundle of X. The spaces J_nX, $n \in \mathbb{Z}_{\geq 0}$ constitute a projective system and the limit $J_\infty X := \varprojlim J_nX$ exists. We can define a measure μ_X and construct an integration theory on $J_\infty X$ with values in some ring (or semiring) in which we can add and multiply the classes $\{V\}$ of varieties V and some class of infinite sums are defined. For example, we can use a completion of the Grothendieck ring of mixed Hodge structures ($k = \mathbb{C}$) or mixed Galois representations (k a finite field). If X is smooth, then we have

$$\int_{J_\infty X} 1 \, d\mu_X = \mu_X(J_\infty X) = \{X\}.$$

To generalize the theory to Deligne-Mumford stacks, it is not sufficient to consider only $R[[t]]/t^{n+1}$-points of a stack. Inspired by a work of Abramovich and Vistoli [AV], the author introduced the notion of twisted jets in [Yas1]. Let $\mathcal{X}$ be a separated Deligne-Mumford stack of finite type over k and $\boldsymbol{\mu}_{l,k}$ be the group scheme of l-th roots of unity for a positive integer l prime to the characteristic of k. A *twisted n-jet* over $\mathcal{X}$ is a representable morphism from a quotient stack $[(\operatorname{Spec} R[[t]]/t^{nl+1})/\boldsymbol{\mu}_{l,k}]$ to $\mathcal{X}$. We will prove that the category $\mathcal{J}_n\mathcal{X}$ of twisted n-jets is a Deligne-Mumford stack. If k is algebraically closed

and $\mathcal{X}$ is a quotient stack $[M/G]$, then we have

$$\mathcal{J}_0\mathcal{X} \cong \coprod_{g\in \mathrm{Conj}(G)} [M^g/C_g].$$

Here $\mathrm{Conj}(G)$ is a representative set of conjugacy classes, M^g the fixed point locus of g and C_g is the centralizer of g. The right hand side often appears in the study of McKay correspondence. There exists also the projective limit $\mathcal{J}_\infty\mathcal{X} := \varprojlim \mathcal{J}_n\mathcal{X}$. When $\mathcal{X}$ is smooth, we define a measure $\mu_\mathcal{X}$ and construct an integration theory on the point set $|\mathcal{J}_\infty\mathcal{X}|$.

Let $\mathbb{L}$ be the class $\{\mathbb{A}^1_k\}$ of an affine line. To a variety X and an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, we can associate a function $\mathrm{ord}\,\mathcal{I} : J_\infty X \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and a function $\mathbb{L}^{\mathrm{ord}\,\mathcal{I}}$. Consider a proper birational morphism $f : Y \to X$ of varieties with Y smooth. The Jacobian ideal sheaf $\mathrm{Jac}_f \subset \mathcal{O}_Y$ is defined to be the 0-th Fitting ideal of $\Omega_{Y/X}$. If X is also smooth, then this is identical with the ideal sheaf of the relative canonical divisor $K_{Y/X} := K_Y - f^*K_X$. Let $f_\infty : J_\infty Y \to J_\infty X$ be the morphism induced by f. The relation of the measures μ_X and μ_Y is described by the following *transformation rule*:

$$\int F d\mu_X = \int (F \circ f_\infty)\mathbb{L}^{-\mathrm{ord}\,\mathrm{Jac}_f} d\mu_Y.$$

This formula was proved by Kontsevich [Kon], Denef and Loeser [DL1], and Sebag [Seb]. Using this, we obtain many results in the birational geometry. For instance, Kontsevich proved the following: If $f : Y \to X$ and $f' : Y \to X'$ are proper birational morphisms of smooth proper varieties over $\mathbb{C}$, and if $K_{Y/X} = K_{Y/X'}$, then the Hodge structure of $H^i(X, \mathbb{Q})$ and that of $H^i(X', \mathbb{Q})$ are isomorphic.

We generalize the transformation rule to Deligne-Mumford stacks. If we consider only representable morphisms, no interesting phenomenon appears. A morphism of Deligne-Mumford stacks is said to be *birational* if it induces an isomorphism of open dense substacks. For example, if M is a variety with an effective action of a finite group G, then the natural morphism from the quotient stack $[M/G]$ to the quotient variety M/G is birational. A morphism $f : \mathcal{Y} \to \mathcal{X}$ is said to be *tame* if for every geometric point y of $\mathcal{Y}$, $\mathrm{Ker}\,(\mathrm{Aut}(y) \to \mathrm{Aut}(f(x)))$ is of order prime to the characteristic of k. The transformation rule is generalized to tame, proper and birational morphisms.

Let $\tilde{x}$ be a geometric point of $\mathcal{J}_0\mathcal{X}$ and x its image in $\mathcal{X}$. A $\boldsymbol{\mu}_l$-action on the tangent space $T_x\mathcal{X}$ derives from $\tilde{x}$. If for suitable basis, $\zeta \in \boldsymbol{\mu}_l$

acts by $\mathrm{diag}(\zeta^{a_1}, \dots, \zeta^{a_d})$, $1 \le a_i \le l$, then we define

$$\mathrm{sht}(\tilde{x}) := d - \frac{1}{l}\sum_{i=1}^{l} a_i.$$

Thus we have a function $\mathrm{sht} : |\mathcal{J}_0\mathcal{X}| \to \mathbb{Q}$. We denote by $\mathfrak{s}_{\mathcal{X}}$, the composite $|\mathcal{J}_\infty\mathcal{X}| \to |\mathcal{J}_0\mathcal{X}| \xrightarrow{\mathrm{sht}} \mathbb{Q}$.

Also for a birational morphism $f : \mathcal{Y} \to \mathcal{X}$ of Deligne-Mumford stacks, we define its Jacobian ideal sheaf, Jac_f, in the same way. However, the associated order function $\mathrm{ord}\,\mathrm{Jac}_f$ is a $\mathbb{Q}$-valued function.

Now we can formulate the generalized transformation rule as follows:

Theorem 1.1. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a tame proper birational morphism of Deligne-Mumford stacks of finite type and pure dimension. Suppose that $\mathcal{Y}$ is smooth and that $\mathcal{X}$ is either a smooth Deligne-Mumford stack or a variety. Then we have*

$$\int F\mathbb{L}^{\mathfrak{s}_{\mathcal{X}}} d\mu_{\mathcal{X}} = \int (F \circ f_\infty)\mathbb{L}^{-\mathrm{ord}\,\mathrm{Jac}_f + \mathfrak{s}_{\mathcal{Y}}} d\mu_{\mathcal{Y}}.$$

(See Theorem 3.41 for details).

Remark 1.2. The theorem was proved in [Yas1] for the morphisms from a smooth stack $\mathcal{X}$ without reflection to its coarse moduli space with Gorenstein singularities.

We apply the transformation rule to the birational geometry of Deligne-Mumford stacks. We recall Batyrev's work [Bat] as a background of this subject. Suppose $k = \mathbb{C}$. Let M be a smooth variety, G a finite group acting effectively on M and $X = M/G$ be the quotient variety. By calculations, Batyrev showed a relation of the orbifold E-function of the G-variety M and the stringy E-function of X. Denef and Loeser [DL2] proved a similar result with motivic integration. From the viewpoint of stacks, the orbifold E-function is defined rather for the quotient stack $[M/G]$ than for the G-variety M. The natural morphism $[M/G] \to X$ is proper and birational. Then Batyrev's result can be viewed as a relation of invariants of birational stacks. We will reformulate his results in a full generality from this viewpoint.

Let $\mathcal{X}$ be a smooth Deligne-Mumford stack of finite type over a perfect field k, $D = \sum u_i D_i$ a $\mathbb{Q}$-divisor on $\mathcal{X}$ and $W \subset |\mathcal{X}|$ a constructible subset. We put $\mathfrak{I}_D := \sum u_i \cdot \mathrm{ord}\,\mathcal{I}_{D_i}$. We define an invariant

$$\Sigma_W(\mathcal{X}, D) := \int_{\pi^{-1}(W)} \mathbb{L}^{\mathfrak{I}_D + \mathfrak{s}_{\mathcal{X}}} d\mu_{\mathcal{X}}.$$

Here $\pi : \mathcal{J}_\infty\mathcal{X} \to \mathcal{X}$ is the natural projection. The function $\mathrm{sht} : |\mathcal{J}_0\mathcal{X}| \to \mathbb{Q}$ is, in fact, locally constant and for a connected component

$\mathcal{V}$, $\mathrm{sht}(\mathcal{V}) \in \mathbb{Q}$ is well-defined. If $D = 0$ and $W = |\mathcal{X}|$, then the invariant is equal to

$$\sum_{\mathcal{V} \subset \mathcal{J}_0 \mathcal{X}} \{\mathcal{V}\} \mathbb{L}^{\mathrm{sht}(\mathcal{V})}.$$

If $k = \mathbb{C}$ and $\mathcal{X}$ is proper, this has the information of the Hodge structure of the orbifold cohomology defined below.

In characteristic zero, we can generalize the invariant $\Sigma_W(\mathcal{X}, D)$ to the case where $\mathcal{X}$ is singular: A *log Deligne-Mumford stack* is defined to be the pair $(\mathcal{X}, D)$ of a normal Deligne-Mumford stack $\mathcal{X}$ of finite type and a $\mathbb{Q}$-divisor D on $\mathcal{X}$ such that $K_{\mathcal{X}} + D$ is $\mathbb{Q}$-Cartier. For a log Deligne-Mumford stack $(\mathcal{X}, D)$ and a constructible subset $W \subset |\mathcal{X}|$, if $f : \mathcal{Y} \to \mathcal{X}$ is a proper birational morphism with $\mathcal{Y}$ smooth, then we define $\Sigma_W(\mathcal{X}, D) := \Sigma_{f^{-1}(W)}(\mathcal{Y}, f^*(K_{\mathcal{X}} + D) - K_{\mathcal{Y}})$. This invariant is a common generalization and refinement of the stringy E-function and the orbifold E-function. By a calculation, we will see that $\Sigma_W(\mathcal{X}, D) \neq \infty$ if and only if $(\mathcal{X}, D)$ is Kawamata log terminal around W (For the definition, see Definition 4.17).

The following is the direct consequence of the transformation rule and viewed as a generalization of Batyrev's result and Denef and Loeser's one.

Theorem 1.3. *Let $(\mathcal{X}, D)$ and $(\mathcal{X}', D')$ be log Deligne-Mumford stacks. Assume that there exist a smooth DM stack $\mathcal{Y}$ and proper birational morphisms $f : \mathcal{Y} \to \mathcal{X}$ and $f' : \mathcal{Y} \to \mathcal{X}$ such that $f^*(K_{\mathcal{X}} + D) = (f')^*(K_{\mathcal{X}'} + D')$ and $f^{-1}(W) = (f')^{-1}(W')$. In positive characteristic, assume in addition that $\mathcal{X}$ and $\mathcal{X}'$ are smooth and that f and f' are tame. Then we have*

$$\Sigma_W(\mathcal{X}, D) = \Sigma_{W'}(\mathcal{X}', D').$$

Remark 1.4. Kawamata [Kaw] obtained a closely related result in terms of the derived category.

Finally we give corollaries of this theorem.

Let $G \subset \mathrm{GL}_d(\mathbb{C})$ be a finite subgroup and $X := \mathbb{C}^d/G$ the quotient variety. For $g \in G$, we define a rational number $\mathrm{age}(g)$ as follows: Let l be the order of g and $\zeta := \exp(2\pi\sqrt{-1}/l)$. If we write $g = \mathrm{diag}(\zeta^{a_i}, \dots, \zeta^{a_d})$, $0 \le a_i \le l-1$, for suitable basis of $\mathbb{C}$, then

$$\mathrm{age}(g) := \frac{1}{l} \sum_{i=1}^{d} a_i.$$

If $g \in \mathrm{SL}_d(\mathbb{C})$, then $\mathrm{age}(g)$ is an integer. The following was called the Homological McKay correspondence. It was proved by Y. Ito and Reid

[IR] for dimension three and by Batyrev for arbitrary dimension [Bat]. (See also [Rei2]).

Corollary 1.5. *Suppose that $G \subset \mathrm{SL}_d(\mathbb{C})$ and that there is a crepant resolution $Y \to X$. For an even integer i, put*

$$n_i := \sharp\{g \in \mathrm{Conj}(G) | \mathrm{age}(g) = i/2\}.$$

Then we have

$$H^i(Y, \mathbb{Q}) \cong \begin{cases} 0 & (i : odd) \\ \mathbb{Q}(-i/2)^{\oplus n_i} & (i : even). \end{cases}$$

Since $X = \mathbb{C}^d/G$ has only quotient singularities, K_X is $\mathbb{Q}$-Cartier and its pull-back by arbitrary morphism is defined. For a resolution $f : Y \to X$ and for each exceptional prime divisor $E \subset Y$, there is a rational number $a(E, X)$ such that

$$K_Y \equiv f^*K_X + \sum_{E \subset Y} a(E, X)E.$$

The *discrepancy* of X is defined to be the infimum of $a(E, X)$ for all resolutions $Y \to X$ and all exceptional divisors $E \subset Y$. The following is a refinement of Reid–Shepherd-Barron–Tai criterion for canonical (or terminal) quotient singularities (see [Rei1, §4.11]).

Corollary 1.6. *For a finite group $G \subset \mathrm{GL}_d(\mathbb{C})$ without reflection, the discrepancy of $X = \mathbb{C}^d/G$ is equal to*

$$\min\{\mathrm{age}(g) | 1 \ne g \in G\} - 1.$$

Chen and Ruan [CR] defined a new cohomology for topological orbifolds (Satake's V-manifolds), called *orbifold cohomology.* We give its algebraic version. Let $\mathcal{X}$ be a smooth Deligne-Mumford stack over $\mathbb{C}$. For $i \in \mathbb{Q}$, we define

$$H^i_{orb}(\mathcal{X}, \mathbb{Q}) := \bigoplus_{\mathcal{V} \subset \mathcal{J}_0\mathcal{X}} H^{i-2\mathrm{sht}(\mathcal{V})}(\bar{\mathcal{V}}, \mathbb{Q}) \otimes \mathbb{Q}(-\mathrm{sht}(\mathcal{V})).$$

Here $\bar{\mathcal{V}}$ is the coarse moduli space of $\mathcal{V}$. If $\mathcal{X}$ is proper, then $H^i_{orb}(\mathcal{X}, \mathbb{Q})$ is a pure Hodge structure of weight i. (We define Hodge structure with fractional weights in the trivial fashion.) The following was conjectured by Ruan [Rua] and a weak version was proved by Lupercio-Poddar [LP] and the author [Yas1] independently. This is a generalization of Kontsevich's theorem stated above.

Corollary 1.7. *Let $\mathcal{X}$ and $\mathcal{X}'$ be proper and smooth Deligne-Mumford stacks of finite type over $\mathbb{C}$. Suppose that there exist a smooth Deligne-Mumford stack $\mathcal{Y}$ and proper birational morphisms $f : \mathcal{Y} \to \mathcal{X}$ and*

$f' : \mathcal{Y} \to \mathcal{X}'$ such that $K_{\mathcal{Y}/\mathcal{X}} = K_{\mathcal{Y}/\mathcal{X}'}$. Then for every $i \in \mathbb{Q}$, there is an isomorphism of Hodge structures

$$H^i_{orb}(\mathcal{X}, \mathbb{Q}) \cong H^i_{orb}(\mathcal{X}', \mathbb{Q}).$$

We also define the p-adic orbifold cohomology. Let $\mathcal{X}$ be a smooth Deligne-Mumford stack over a finite field k. and p a prime number different from the characteristic of k. If necessary, replacing k with its finite extension, we define

$$H^i_{orb}(\mathcal{X} \otimes \bar{k}, \mathbb{Q}_p) := \bigoplus_{\mathcal{V} \subset \mathcal{J}_0 \mathcal{X}} H^{i-2\mathrm{sht}(\mathcal{V})}(\bar{\mathcal{V}} \otimes \bar{k}, \mathbb{Q}_p) \otimes \mathbb{Q}_p(-\mathrm{sht}(\mathcal{V})).$$

Replacing k is necessary to ensure that fractional Tate twists $\mathbb{Q}_p(-\mathrm{sht}(\mathcal{V}))$ exist.

Corollary 1.8. *Let $\mathcal{X}$ and $\mathcal{X}'$ be proper and smooth Deligne-Mumford stacks of finite type over a finite field k. Suppose that there exist a smooth Deligne-Mumford stack $\mathcal{Y}$ and tame proper birational morphisms $f : \mathcal{Y} \to \mathcal{X}$ and $f' : \mathcal{Y} \to \mathcal{X}'$ such that $K_{\mathcal{Y}/\mathcal{X}} = K_{\mathcal{Y}/\mathcal{X}'}$. Suppose that the p-adic orbifold cohomology groups of $\mathcal{X}$ and $\mathcal{X}'$ are defined. Then for every $i \in \mathbb{Q}$, there is an isomorphism of Galois representations*

$$H^i_{orb}(\mathcal{X} \otimes_k \bar{k}, \mathbb{Q}_p)^{ss} \cong H^i_{orb}(\mathcal{X}' \otimes_k \bar{k}, \mathbb{Q}_p)^{ss}.$$

Here the superscript "ss" means the semisimplification.

For varieties, T. Ito [Ito1] and Wang [Wan] obtained a similar result over number fields.

1.1. **Notation and convention.** Throughout this paper, we work over a perfect base field k. A Deligne-Mumford stack (DM stack for short) is supposed to be separated. What we mean by a variety is a separated algebraic space of finite type over k.

- $\mathbb{N}$, $\mathbb{Z}_{\geq 0}$: the set of positive integers and that of non-negative integers
- $[M/G]$: quotient stack
- $|\mathcal{X}|$: the set of points of $\mathcal{X}$
- $\bar{\mathcal{X}}$: the coarse moduli space of a DM stack $\mathcal{X}$
- $\mathcal{D}^l_{n,S} := [D_{nl,S}/\boldsymbol{\mu}_{l,k}]$
- $D_{n,S} := \operatorname{Spec} R[[t]]/t^{n+1}$ $(S = \operatorname{Spec} R)$
- $\boldsymbol{\mu}_l \subset \bar{k}$: the group of l-th roots of unity
- $\boldsymbol{\mu}_{l,k} := \operatorname{Spec} k[x]/(x^l - 1)$: the group scheme of l-th roots of unity over k
- $\mathrm{Conj}(G)$: a representative set of conjugacy classes $[g]$ of $g \in G$

- $\mathrm{Conj}(\boldsymbol{\mu}_l, G)$: a representative set of conjugacy classes of $\boldsymbol{\mu}_l \hookrightarrow G$
- $J_n X$: n-jet space
- $J_n^{(a)} X$: For a scheme with G-action and $a : \boldsymbol{\mu}_l \hookrightarrow G$, $J_n^{(a)} X \subset J_n X$ is the locus where the two $\boldsymbol{\mu}_l$-actions on $J_n X$ coincide
- $\mathcal{J}_n^l \mathcal{X}$: the stack of twisted n-jets of order l
- $\mathcal{J}_n \mathcal{X} := \coprod_{\mathrm{char}\,(k) \nmid l} \mathcal{J}_n^l \mathcal{X}$: the stack of twisted n-jets
- $\pi_n : \mathcal{J}_\infty \mathcal{X} \to \mathcal{J}_n \mathcal{X}$, $\pi : \mathcal{J}_\infty \mathcal{X} \to \mathcal{X}$: natural projections
- $f_n : \mathcal{J}_n \mathcal{Y} \to \mathcal{J}_n \mathcal{X}$: the morphism induced by $f : \mathcal{Y} \to \mathcal{X}$
- $\mathfrak{R}$, $\mathfrak{S}$: the semirings of equivalence classes of convergent stacks and convergent spaces
- $\mathbb{L} := \{\mathbb{A}_k^1\}$
- MHS and $MHS^{1/r}$: the category of mixed Hodge structures and the category of $\frac{1}{r}\mathbb{Z}$-indexed ones
- $\mathbf{G}_k := \mathrm{Gal}(\bar{k}/k)$: absolute Galois group
- $MR(\mathbf{G}_k, \mathbb{Q}_p)$: the category of mixed Galois representations
- $\mu_{\mathcal{X}}$: motivic measure
- $\mathrm{sht}(p)$, $\mathrm{sht}(\mathcal{V})$: shift number
- $\mathfrak{s}_{\mathcal{X}}$: the composite $\mathcal{J}_\infty \mathcal{X} \xrightarrow{\pi_0} \mathcal{J}_0 \mathcal{X} \xrightarrow{\mathrm{sht}} \mathbb{Q}$
- Jac_f, Jac_X : the Jacobian ideal sheaves of a morphism f and a variety X
- $\mathrm{ord}\,\mathcal{I}$: the order function of an ideal sheaf $\mathcal{I}$ over $\mathcal{J}_\infty \mathcal{X}$
- $\mathfrak{I}_D$: For a $\mathbb{Q}$-divisor $D = \sum u_i D_i$, if $\mathcal{I}_{D_i}$ is the ideal sheaf of D_i, then $\mathfrak{I}_D := \sum u_i \mathrm{ord}\,\mathcal{I}_{D_i}$.
- $\omega_{\mathcal{X}}$, $K_{\mathcal{X}}$ and $K_{\mathcal{Y}/\mathcal{X}}$: canonical sheaf, canonical divisor and relative canonical divisor

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2. Stacks of twisted jets

2.1. Short review of the Deligne-Mumford stacks.

2.1.1. We first review the Deligne-Mumford (DM) stack very briefly. We mention the book of Laumon and Moret-Bailly [LMB] as a reference of stacks. We will sometimes use results from it.

Fix a base field k. Let (Aff/k) be the category of affine schemes over k. A DM stack $\mathcal{X}$ is a category equipped with a functor $\mathcal{X} \to (\mathrm{Aff}/k)$ which satisfies several conditions. It should be a fibered category over (Aff/k) and is usually best understood in terms of the fiber categories $\mathcal{X}(S)$, for $S \in (\mathrm{Aff}/k)$, and the pull-back functors $f^* : \mathcal{X}(T) \to \mathcal{X}(S)$ for $f : S \to T$. The $\mathcal{X}(S)$ are groupoids with, at least for S of finite type, finite automorphism groups.

The DM stacks constitute a 2-category. In terms of the fiber categories, a 1-morphism (or simply morphism) $f : \mathcal{Y} \to \mathcal{X}$ is the data of functors $f_S : \mathcal{Y}(S) \to \mathcal{X}(S)$, compatible with pull-backs, and a 2-morphism $f \to g$ is a system of morphisms of functors $f_S \to g_S$, compatible with pull-backs. A scheme, or more generally an algebraic space X is identified with the DM stack with fibers the discrete categories with sets of objects the $X(S) := \mathrm{Hom}(S, X)$. A diagram of stacks

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[dr, "h"'] & \mathcal{Y} \arrow[d, "g"] \\ & \mathcal{Z} \end{tikzcd}$$

is said to be commutative if a 2-isomorphism $g \circ f \cong h$ has been given. The strict identity $g \circ f = h$ is not required.

A morphism $f : \mathcal{Y} \to \mathcal{X}$ of DM stacks is called *representable* if for every morphism $M \to \mathcal{X}$ with M an algebraic space, the fiber product $M \times_{\mathcal{X}} \mathcal{Y}$ is also an algebraic space. It is equivalent to that for every object $\xi \in \mathcal{Y}$, the natural map $\mathrm{Aut}(\xi) \to \mathrm{Aut}(f(\xi))$ is injective. We can generalize many properties of a morphism of schemes to DM stack; étale, smooth, proper etc. By a condition in the definition, for every DM stack $\mathcal{X}$, there exist an algebraic space M and an étale surjective morphism $M \to \mathcal{X}$, which is called an *atlas*. We say that $\mathcal{X}$ is smooth, normal etc if an atlas is so.

The diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ of a DM stack $\mathcal{X}$ is , by definition, representable. We say that $\mathcal{X}$ is *separated* if Δ is finite, that is, quasi-finite and proper. Note that Δ is not immersion unless $\mathcal{X}$ is an algebraic space. In this paper, every DM stack is supposed to be separated.

2.1.2. *Points and coarse moduli space.* A *point* of a DM stack $\mathcal{X}$ is an equivalence class of morphisms $\mathrm{Spec}\, K \to \mathcal{X}$ with $K \supset k$ a field by the following equivalence relation; morphisms $\mathrm{Spec}\, K_1 \to \mathcal{X}$ and

$\operatorname{Spec} K_2 \to \mathcal{X}$ are equivalent if there is another field $K_3 \supset K_1, K_2 \supset k$ making the following diagram commutative.

$$\begin{tikzcd} \operatorname{Spec} K_3 \arrow[r] \arrow[d] & \operatorname{Spec} K_2 \arrow[d] \\ \operatorname{Spec} K_1 \arrow[r] & \mathcal{X} \end{tikzcd}$$

We denote by the set of the points by $|\mathcal{X}|$. It carries a Zariski topology; $A \subset |\mathcal{X}|$ is an open subset if $A = |\mathcal{Y}|$ for some open immersion $\mathcal{Y} \hookrightarrow \mathcal{X}$. (see [LMB] for details). If X is a scheme, then $|X|$ is equal to the underlying topological space as sets.

A coarse moduli space of a DM stack $\mathcal{X}$ is an algebraic space equipped with a morphism $\mathcal{X} \to X$ such that every morphism $\mathcal{X} \to Y$ with Y algebraic space uniquely factors through X and for every algebraically closed field $K \supset k$, the map $\mathcal{X}(K)/isom \to X(K)$ is bijective. By the definition, it is clear that the coarse moduli space is unique up to isomorphism. Keel and Mori [KM] proved that for every DM stack, the coarse moduli space exists. If X is the coarse moduli space of $\mathcal{X}$, then the map $|\mathcal{X}| \to |X|$ is a homeomorphism.

2.1.3. *Quotient stack.* One of the simplest examples is the quotient stack. Let M be an algebraic space and G a finite group (or an étale finite group scheme over k) acting on M. Then we can define the quotient stack $[M/G]$ as follows; an object over a scheme S is a pair of a G-torsor $P \to S$ and a G-equivariant morphism $P \to M$ and a morphism of $(P \to S, P \to M)$ to $(Q \to T, Q \to M)$ over a morphism $S \to T$ is a G-equivariant morphism $P \to Q$ compatible with the other morphisms. This stack has the canonical atlas $M \to [M/G]$. There is also a natural morphism $[M/G] \to M/G$ which makes M/G the coarse moduli space. The composition $M \to [M/G] \to M/G$ is the quotient map.

2.2. **Stacks of twisted jets.**

2.2.1. In the article [Yas1], the author introduced the notion of twisted jets. There, only twisted jets over fields were considered and the stack of twisted jets was constructed as a closed substack of another stack. By contrast, in this paper, we consider the category of twisted jets parameterized by arbitrary affine scheme and verify that it is actually a DM stack.

We first recall jets and arcs over a variety. Here we mean a separated algebraic space of finite type by a variety. Let X be a variety and n a

non-negative integer. The functor

$$\begin{aligned}(\mathrm{Aff}/k) &\to (\mathrm{Sets})\\ \operatorname{Spec} R &\mapsto \operatorname{Hom}(\operatorname{Spec} R[[t]]/t^{n+1}, X)\end{aligned}$$

is representable by a variety J_nX, called the *n-jet space*. The natural surjection $k[[t]]/t^{n+2} \twoheadrightarrow k[[t]]/t^{n+1}$ induces a natural projection $J_{n+1}X \to J_nX$. Since they are all affine morphisms, the projective limit $J_\infty X := \varprojlim J_nX$ exists. This is an algebraic space, but not generally of finite type. We call this the *arc space*. For every field extension $K \supset k$, there is an identification

$$\operatorname{Hom}(\operatorname{Spec} K, J_\infty X) = \operatorname{Hom}(\operatorname{Spec} K[[t]], X).$$

An *arc* of X is a point of $J_\infty X$, that is, a morphism $\operatorname{Spec} K[[t]] \to X$.

For $S = \operatorname{Spec} R \in (\mathrm{Aff}/k)$ and a non-negative integer n, we put

$$D_{n,S} := \operatorname{Spec} R[[t]]/t^{n+1}.$$

For l a positive integer prime to the characteristic of k, we denote by $\boldsymbol{\mu}_l \subset \bar{k}$ the cyclic group of l-th roots of unity. We define also the group scheme of l-th roots of unity over k

$$\boldsymbol{\mu}_{l,k} := \operatorname{Spec} k[x]/(x^l - 1).$$

When $\boldsymbol{\mu}_{l,k}$ is a constant group scheme, then we identify it with the group $\boldsymbol{\mu}_l$. The natural action of $\boldsymbol{\mu}_{l,k}$ on $D_{n,S}$ is defined by $t \mapsto x \otimes t$. We put

$$\mathcal{D}^l_{n,S} := [D_{nl,S}/\boldsymbol{\mu}_{l,k}].$$

Also for $n = \infty$, and for a field $K \supset k$, we put

$$D_{\infty,K} := \operatorname{Spec} K[[t]] \text{ and } \mathcal{D}^l_{\infty,K} := [D_{\infty,K}/\boldsymbol{\mu}_{l,k}].$$

Definition 2.1. Let $\mathcal{X}$ be a DM stack. A *twisted n-jet of order l* of $\mathcal{X}$ over S is a representable morphism $\mathcal{D}^l_{n,S} \to \mathcal{X}$. For a field $K \supset k$, a *twisted arc (or twisted infinite jet) of order l* of $\mathcal{X}$ over K is a representable morphism $\mathcal{D}^l_{\infty,K} \to \mathcal{X}$.

Definition 2.2. Let $\mathcal{X}$ be a DM stack. Suppose $n < \infty$. We define the *stack of twisted n-jets of order l*, denoted $\mathcal{J}^l_n\mathcal{X}$, as follows; an object over $S \in (\mathrm{Aff}/k)$ is a representable morphism $\mathcal{D}^l_{n,S} \to \mathcal{X}$, a morphism from $\gamma : \mathcal{D}^l_{n,S} \to \mathcal{X}$ to $\gamma' : \mathcal{D}^l_{n,T} \to \mathcal{X}$ over $f : S \to T$ is a 2-morphism from γ to $f' \circ \gamma'$, where $f' : \mathcal{D}^l_{n,S} \to \mathcal{D}^l_{n,T}$ is the morphism naturally induced by f.

We will prove that it is actually a DM stack.

Definition 2.3. We define the *stack of twisted n-jet* of $\mathcal{X}$ by

$$\mathcal{J}_n\mathcal{X} := \coprod_{\mathrm{char}\,(k)\nmid l} \mathcal{J}_n^l\mathcal{X}.$$

If $\mathcal{X}$ is of finite type, then $\mathcal{J}_n^l\mathcal{X}$ is empty for sufficiently large l and $\mathcal{J}_n\mathcal{X}$ is in fact the disjoint sum of only finitely many $\mathcal{J}_n^l\mathcal{X}$.

Lemma 2.4. *The category $\mathcal{J}_n^l\mathcal{X}$ is a stack.*

Proof. For an object $\gamma : \mathcal{D}_{n,S}^l \to \mathcal{X}$ of $\mathcal{J}_n^l\mathcal{X}$ and for a morphism $f : T \to S$, we have a"pull-back", $\gamma_T := f' \circ \gamma$ which is unique up to 2-isomorphisms. Here $f' : \mathcal{D}_{n,T}^l \to \mathcal{D}_{n,S}^l$ is the natural morphism induced by f. Hence $\mathcal{J}_n^l\mathcal{X}$ is a groupoid.

We first show that for two objects $\gamma, \gamma' : \mathcal{D}_{n,S}^l \to \mathcal{X}$, the functor

$$\begin{aligned} \mathcal{I}som(\gamma, \gamma') : (\mathrm{Aff}/S) &\to (\mathrm{Sets}) \\ (T \to S) &\mapsto \mathrm{Hom}_{(\mathcal{J}_n^l\mathcal{X})(T)}(\gamma_T, \gamma'_T). \end{aligned}$$

is a sheaf. Consider a morphism $T \to S$ and an étale cover $\coprod T_i \to T$. Let $T_{ij} := T_i \times_T T_j$. For every object α of $\mathcal{D}_{n,T}^l$, we have the pull-backs α_i and α_{ij} to $\mathcal{D}_{n,T_i}^l$ and $\mathcal{D}_{n,T_{ij}}^l$ respectively. Since $\mathcal{X}$ is a prestack, the sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{X}(T)}(\gamma_T(\alpha), \gamma'_T(\alpha)) &\to \coprod \mathrm{Hom}_{\mathcal{X}(T_i)}(\gamma_{T_i}(\alpha_i), \gamma'_{T_i}(\alpha_i)) \\ &\rightrightarrows \coprod \mathrm{Hom}_{\mathcal{X}(T_{ij})}(\gamma_{T_{ij}}(\alpha_{ij}), \gamma'_{T_{ij}}(\alpha_{ij})) \end{aligned}$$

is exact. Since a morphism of twisted jets is a natural transformation of functors, it implies that the sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{(\mathcal{J}_n^l\mathcal{X})(T)}(\gamma_T, \gamma'_T) &\to \coprod \mathrm{Hom}_{(\mathcal{J}_n^l\mathcal{X})(T_i)}(\gamma_{T_i}, \gamma'_{T_i}) \\ &\rightrightarrows \coprod \mathrm{Hom}_{(\mathcal{J}_n^l\mathcal{X})(T_{ij})}(\gamma_{T_{ij}}, \gamma'_{T_{ij}}) \end{aligned}$$

is also exact, and the functor $\mathcal{I}som(\gamma, \gamma')$ is a sheaf.

It remains to show that one can glue objects. Let $\coprod T_i \to T$ be an étale cover, let $\gamma_i : \mathcal{D}_{n,T_i}^l \to \mathcal{X}$ be twisted jets and let $h_{ij} : (\gamma_i)_{T_{ij}} \to (\gamma_j)_{T_{ij}}$ be a morphism in $(\mathcal{J}_n^l\mathcal{X})(T_{ij})$. Assume that they satisfy the cocycle condition. Then for every object α of $\mathcal{D}_{n,T}^l$, we can glue the objects $\gamma_i(\alpha_i)$ of $\mathcal{X}$, because $\mathcal{X}$ is a stack. Therefore we can determine the image of α and obtain a functor $\gamma : \mathcal{D}_{n,T}^l \to \mathcal{X}$ which is clearly representable. Thus we have verified all conditions. $\square$

2.2.2. Let $\mathcal{Y} \to \mathcal{X}$ be a representable morphism of DM stacks. Then for a twisted jet $\mathcal{D}^l_{n,S} \to \mathcal{Y}$, composing the morphisms, we obtain a twisted jet $\mathcal{D}^l_{n,S} \to \mathcal{X}$. Thus we have a natural morphism $\mathcal{J}^l_n\mathcal{Y} \to \mathcal{J}^l_n\mathcal{X}$.

In [Yas1], we defined a *barely faithful morphism* to be a morphism $f : \mathcal{Y} \to \mathcal{X}$ of DM stacks such that for every object ξ of $\mathcal{Y}$, the map $\mathrm{Aut}(\xi) \to \mathrm{Aut}(f(\xi))$ is bijective. Thus all barely faithful morphisms are representable. Barely faithful morphisms are stable under base change [Yas1, Lemma 4.21].

Lemma 2.5. *Let $\mathcal{Y} \to \mathcal{X}$ be a barely faithful and formally étale morphism of DM stacks. Then the naturally induced diagram*

$$\begin{tikzcd}
\mathcal{J}^l_n\mathcal{Y} \arrow[r] \arrow[d] & \mathcal{J}^l_n\mathcal{X} \arrow[d] \\
\mathcal{Y} \arrow[r] & \mathcal{X}
\end{tikzcd}$$

is cartesian.

Proof. Consider a commutative diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\
\mathcal{D}^l_{n,S} \arrow[r] & \mathcal{X}
\end{tikzcd}$$

where the bottom arrow is representable and the left arrow is a natural one. Then we claim that there exists a unique morphism $\mathcal{D}^l_{n,S} \to \mathcal{Y}$ which fits into the diagram. The lemma easily follows from it.

Without loss of generality, we can assume that S is connected. Let $\mathcal{U} \subset \mathcal{D}^l_{n,S} \times_{\mathcal{X}} \mathcal{Y}$ be the connected component containing the image of S. Then the natural morphism $\mathcal{U} \to \mathcal{D}^l_{n,S}$ is barely faithful, formally étale and bijective, hence an isomorphism. It shows our claim. Thus we obtain an equivalence of categories, $\mathcal{Y} \times_{\mathcal{X}} \mathcal{J}^l_n\mathcal{X} \cong \mathcal{J}^l_n\mathcal{Y}$. $\square$

For every DM stack $\mathcal{X}$, there are finite groups G_i, schemes M_i with G_i-action and a morphism $\coprod_i [M_i/G_i] \to \mathcal{X}$ which is étale, surjective and barely faithful. Hence thanks to Lemma 2.5, in proving that $\mathcal{J}^l_n\mathcal{X}$ is a DM stack, we may assume that $\mathcal{X}$ is a quotient stack $[M/G]$. Let k'/k be the field extension by adding all l-th roots of unity for the order l of elements of G prime to the characteristic of k. Replacing k with k' and M with $M \otimes_k k'$, we may assume that $\boldsymbol{\mu}_{l,k}$ is a constant group scheme for l such that there is a twisted jet $\mathcal{D}^l_{n,S} \to [M/G]$. The action $\boldsymbol{\mu}_l \curvearrowright D_{n,S}$ induces an action $\boldsymbol{\mu}_l \curvearrowright J_nM$. On the other hand, for each embedding $a : \boldsymbol{\mu}_l \hookrightarrow G$, $\boldsymbol{\mu}_l$ acts on M as a subgroup of G and on J_nM.

Definition 2.6. We define $J_n^{(a)}M$ to be the closed subscheme of J_nM where the two actions $\boldsymbol{\mu}_l \curvearrowright J_nM$ are identical.

Definition 2.7. We define $\mathrm{Conj}(\boldsymbol{\mu}_l, G)$ to be a representative set of the conjugacy classes of embeddings $\boldsymbol{\mu}_l \hookrightarrow G$.

Proposition 2.8. *For $0 \le n \le \infty$, there is an isomorphism*

$$\mathcal{J}_n^l \mathcal{X} \cong \coprod_{a \in \mathrm{Conj}(\boldsymbol{\mu}_l, G)} [J_{nl}^{(a)}M/C_a].$$

Here C_a is the centralizer of a. By this isomorphism, $[J_{nl}^{(a)}M/C_a]$ corresponds to twisted jets $\mathcal{D}_{n,S}^l \to \mathcal{X}$ inducing $a : \boldsymbol{\mu}_l \hookrightarrow G$.

Proof. Let $m := nl$. Choose a primitive l-th root $\zeta \in \boldsymbol{\mu}_l$ of unity. Let $\gamma : \mathcal{D}_{n,S}^l \to \mathcal{X}$ be an object over S of $\mathcal{J}_n^l\mathcal{X}$. The canonical atlas $D_{m,S} \to \mathcal{D}_{n,S}^l$ corresponds to the object α of $\mathcal{D}_{n,S}^l$

$$\begin{tikzcd} D_{m,S} \times \boldsymbol{\mu}_l \arrow[r, "\boldsymbol{\mu}_l\text{-action}"] \arrow[d, "\text{trivial } \boldsymbol{\mu}_l\text{-torsor}"'] & D_{m,S} \\ D_{m,S} & \end{tikzcd} .$$

The morphism

$$\theta := \zeta \times \zeta^{-1} : D_{m,S} \times \boldsymbol{\mu}_l \to D_{m,S} \times \boldsymbol{\mu}_l$$

is an automorphism of α *over* $\zeta : D_{m,S} \to D_{m,S}$, whose order is l. Any other object of $\mathcal{D}_{n,S}^l$ is a pull-back of α and any automorphism is a pull-back of a power of θ. Therefore the twisted jet γ is determined by the images of α and θ in $\mathcal{X}$.

Let the diagram

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, "p"'] & M \\ D_{m,S} & \end{tikzcd}$$

be the object over $D_{m,S}$ of $\mathcal{X}$ which is the image of α by γ. Let λ be its automorphism over $\zeta : D_{m,S} \to D_{m,S}$ which is the image of θ. Because γ is representable, the order of λ is also l. Let $Q := P \times_{D_{m,S}} S$. Then $P \to D_{m,S}$ is isomorphic as torsors to $D_{m,k} \times_k Q \to D_{m,k} \times_k S$. Since we have chosen a primitive l-th root ζ, we can identify $\mathrm{Conj}(\boldsymbol{\mu}_l, G)$ with a representative set $\mathrm{Conj}^l(G)$ of the conjugacy classes of *elements* of order l.

Claim: *If S is connected, then there are open and closed subsets $Q' \subset Q$ and $P' \subset P$ which, for some $g \in \mathrm{Conj}^l(G)$, are stable under C_g-action and C_g-torsors over S.*

Take an étale cover $T \to S$ such that $Q_T := Q \times_S T$ is isomorphic to the trivial G-torsor $T \times G \to T$ with a right action. Then the pull-back of the automorphism λ is a left action of some $g^{-1} \in G$ over each connected component of T. If necessary, replacing the isomorphism $Q_T \cong T \times G$, we can assume that the automorphism is given by unique $g^{-1} \in \mathrm{Conj}^l(G)$. Let $\phi : T \times G \to Q$ be the natural morphism. Then we see that $\phi(T \times C_g) \cap \phi(T \times (G \setminus C_g)) = \emptyset$, as follows: Let $a \in C_g$, $b \in G \setminus C_g$, $x \in T \times C_g$ and $y \in T \times (G \setminus C_g)$. If $\phi(x) = \phi(y)$, then

$$\phi(x) = \phi(gxg^{-1}) = \lambda\phi(x)g^{-1} = \lambda\phi(y)g^{-1} = \phi(gyg^{-1}) \neq \phi(y).$$

It is a contradiction. Similarly P decomposes also.

Since $(h \circ \lambda)|_{P'} = h|_{P'}$ and $(h \circ g)|_{P'} = (g \circ h)|_{P'}$, we have

$$h \circ (\zeta \times \mathrm{id}_{Q'}) = h \circ (\lambda \circ g)|_{P'} = (g \circ h)|_{P'}.$$

It means that the morphism $D_{m,k} \times_k Q' \to M$ corresponds to a morphism $Q' \to (J_m M)^{\zeta \circ g^{-1}}$ and that the morphism $Q' \to (J_m M)^{\zeta \circ g^{-1}}$ and a C_g-torsor $Q' \to S$ determine an object over S of a quotient stack $[(J_m M)^{\zeta \circ g^{-1}}/C_g]$. Note that $(J_m M)^{\zeta \circ g^{-1}} = J_m^{(a)} M$. Thus we have a morphism $\mathcal{J}_n^l \mathcal{X} \to \coprod [J_m^{(a)} M/C_a]$. The inverse morphism can be constructed by following the argument conversely. □

Theorem 2.9. *Let $\mathcal{X}$ be a DM stack.*

1. *For $n \in \mathbb{Z}_{\geq 0}$, $\mathcal{J}_n^l \mathcal{X}$ and $\mathcal{J}_n \mathcal{X}$ are DM stacks.*
2. *If $\mathcal{X}$ is of finite type (resp. smooth), then for $n \in \mathbb{Z}_{\geq 0}$, then $\mathcal{J}_n^l \mathcal{X}$ and $\mathcal{J}_n \mathcal{X}$ are also of finite type (resp. smooth).*
3. *For every $m \geq n$, the natural projection $\mathcal{J}_m \mathcal{X} \to \mathcal{J}_n \mathcal{X}$ is an affine morphism.*

Proof. 1: There is an étale, surjective and barely faithful morphism $\coprod_i [M/G_i] \to \mathcal{X}$ such that each M_i is a scheme and G_i is a finite group. From Lemma 2.5, Proposition 2.8 and [LMB, Lemme 4.3.3], the morphism $\mathcal{J}_n^l \mathcal{X} \to \mathcal{X}$ is representable. From [LMB, Proposition 4.5], $\mathcal{J}_n^l \mathcal{X}$ is a DM stack. $\mathcal{J}_n \mathcal{X}$ is also a DM stack. The morphism $\mathcal{J}_n^l \mathcal{X} \to \mathcal{X}$ is also separated and so is $\mathcal{J}_n^l \mathcal{X}$.

2 and 3: These also result from Lemma 2.5 and Proposition 2.8. □

In general, a projective system $\{\mathcal{X}_i, \rho_i : \mathcal{X}_{i+1} \to \mathcal{X}_i\}_{i \geq 0}$ of DM stacks such that every ρ_i is representable and affine, there exists a projective limit $\mathcal{X}_\infty = \varprojlim \mathcal{X}_i$. In fact, for each i, there is an $\mathcal{O}_{\mathcal{X}_0}$-algebra $\mathcal{A}_i$ such that $\mathcal{X}_i \cong \operatorname{Spec} \mathcal{A}_i$ (see [LMB, §14.2]) and the $\mathcal{A}_i$'s constitute an

inductive system. We can see that $\mathcal{X}_\infty := \mathrm{Spec}\,(\varinjlim \mathcal{A}_i)$ is the projective limit of the given projective system.

From Theorem 2.9, the projective system $\{\mathcal{J}_n\mathcal{X}\}_n$ (resp. $\{\mathcal{J}^l_n\mathcal{X}\}_n$) has the projective limit

$$\mathcal{J}_\infty\mathcal{X} := \varprojlim \mathcal{J}_n\mathcal{X} \ (\textit{resp.}\ \ \mathcal{J}^l_\infty\mathcal{X} := \varprojlim \mathcal{J}^l_n\mathcal{X}).$$

Then the point set $|\mathcal{J}_\infty\mathcal{X}|$ is identified with the set of the equivalence classes of the twisted arcs $\mathcal{D}^l_{\infty,K} \to \mathcal{X}$ with respect to the following equivalent relation: Let $\gamma_i : \mathcal{D}^l_{\infty,K_i} \to \mathcal{X}$, $i = 1, 2$, be twisted arcs. If for a field $K_3 \supset K_1, K_2$ and natural morphisms $\mathcal{D}^l_{\infty,K_3} \to \mathcal{D}^l_{\infty,K_1}, \mathcal{D}^l_{\infty,K_2}$, the diagram

$$\begin{tikzcd} \mathcal{D}^l_{\infty,K_3} \arrow[r] \arrow[d] & \mathcal{D}^l_{\infty,K_1} \arrow[d, "\gamma_1"] \\ \mathcal{D}^l_{\infty,K_2} \arrow[r, "\gamma_2"'] & \mathcal{X} \end{tikzcd}$$

is commutative, then γ_1 and γ_2 are equivalent.

Remark 2.10. For two stacks $\mathcal{X}$ and $\mathcal{Y}$, we can define a Hom-stack $\mathcal{H}om(\mathcal{X},\mathcal{Y})$ which parameterizes morphisms from $\mathcal{X}$ to $\mathcal{Y}$, and its substack $\mathcal{H}om^{rep}(\mathcal{X},\mathcal{Y})$ which parameterizes representable morphisms. Olsson [Ols] proved that if $\mathcal{X}$ and $\mathcal{Y}$ are Deligne-Mumford stacks satisfying certain conditions, then $\mathcal{H}om(\mathcal{X},\mathcal{Y})$ is a Deligne-Mumford stack. and $\mathcal{H}om^{rep}(\mathcal{X},\mathcal{Y})$ is its open substack. Then, Aoki [Aok] proved that $\mathcal{H}om(\mathcal{X},\mathcal{Y})$ is an Artin stack if $\mathcal{X}$ and $\mathcal{Y}$ are Artin stacks satisfying certain conditions. The stack $\mathcal{J}^l_n\mathcal{X}$ of twisted n-jets of order l ($n < \infty$) is identical with $\mathcal{H}om^{rep}(\mathcal{D}^l_n, \mathcal{X})$.

2.2.3. *Inertia stack.*

Definition 2.11. To each DM stack $\mathcal{X}$, we associate the *inertia stack* $I\mathcal{X}$ defined as follows; an object of $I\mathcal{X}$ is a pair (x, α) with x an object of $\mathcal{X}$ and $\alpha \in \mathrm{Aut}(x)$ and a morphism $(x, \alpha) \to (y, \beta)$ in $I\mathcal{X}$ is a morphism $\phi : x \to y$ in $\mathcal{X}$ with $\phi\alpha = \beta\phi$.

It is known that $I\mathcal{X}$ is isomorphic to $\mathcal{X} \times_{\Delta,\mathcal{X}\times\mathcal{X},\Delta} \mathcal{X}$, where $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is the diagonal morphism. Then the forgetting morphism $I\mathcal{X} \to \mathcal{X}$ is isomorphic to the first projection $\mathcal{X} \times_{\Delta,\mathcal{X}\times\mathcal{X},\Delta} \mathcal{X} \to \mathcal{X}$. Since we have supposed that $\mathcal{X}$ is separated, the diagonal morphism is finite and unramified. Hence the forgetting morphism $I\mathcal{X} \to \mathcal{X}$ is so as well.

Definition 2.12. Let l be a positive integer prime to $\mathrm{char}\,(k)$. We define $I^l\mathcal{X} \subset I\mathcal{X}$ to be the open and closed substack of objects (x, α) such that the order of α is l.

Proposition 2.13. *Assume that k contains all l-th roots of unity. Then for each choice of a primitive l-th root ζ of unity, there is a natural isomorphism $\mathcal{J}_0^l\mathcal{X} \cong I^l\mathcal{X}$.*

Proof. The assertion follows from the fact that giving a representable morphism $\mathcal{D}_0^l \times S \to \mathcal{X}$ is equivalent to giving an object x over S of $\mathcal{X}$ and an embedding $\boldsymbol{\mu}_l \hookrightarrow \mathrm{Aut}(x)$, which is equivalent to giving the image of $\zeta \in \boldsymbol{\mu}_l$. □

The inertia stack is the algebraic counterpart of the twisted sector of an analytic orbifold, which was used to define the orbifold cohomology in [CR]. Since for $k = \mathbb{C}$, there is a canonical choice $\exp(2\pi\sqrt{-1}/l)$ of a primitive l-th root of unity, we have a natural isomorphism $\mathcal{J}_0\mathcal{X} \cong I\mathcal{X}$.

2.3. **Morphism of stacks of twisted jets.** As we saw above, for a representable morphism $\mathcal{Y} \to \mathcal{X}$ of DM stacks, we have a naturally induced morphism $\mathcal{J}_n^l\mathcal{Y} \to \mathcal{J}_n^l\mathcal{X}$. For a morphism $\mathcal{X} \to X$ from a DM stack to an algebraic variety, we can associate a morphism $\mathcal{J}_n^l\mathcal{X} \to J_nX$ as follows: For a twisted jet $\mathcal{D}_{n,S}^l \to \mathcal{X}$, consider the composition $\mathcal{D}_{n,S}^l \to \mathcal{X} \to X$. From the universality of the coarse moduli space, it uniquely factors as $\mathcal{D}_{n,S}^l \to D_{n,S} \to X$ up to isomorphism of $D_{n,S}$. Putting the condition that $\mathcal{D}_{n,S}^l = [D_{nl,S}/\boldsymbol{\mu}_{l,k}] \to D_{n,S}$ is defined by $t \mapsto t^l$, we obtain a unique jet on X.

We generalize these to a general (not necessarily representable) morphism of DM stacks.

Proposition 2.14. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of DM stacks. Then for $0 \le n \le \infty$, we have a natural morphism $f_n : \mathcal{J}_n\mathcal{Y} \to \mathcal{J}_n\mathcal{X}$.*

Proof. We may assume $n < \infty$. Let $\mathcal{D}_{n,S}^l \to \mathcal{Y}$ be an object over a scheme S of $\mathcal{J}_n^l\mathcal{Y}$. Then the composite $\mathcal{D}_{n,S}^l \to \mathcal{Y} \to \mathcal{X}$ is not in general representable. Take the canonical decomposition $\mathcal{D}_{n,S}^l \to \mathcal{E} \to \mathcal{X}$ as in the following lemma. If S is connected, $\mathcal{E}$ must be isomorphic to $\mathcal{D}_{n,S}^{l'}$ for some divisor l' of l. We choose the isomorphism $\mathcal{E} \cong \mathcal{D}_{n,S}^{l'}$ so that the morphism $\mathcal{D}_{n,S}^l \to \mathcal{E} \cong \mathcal{D}_n^{l'}$ induces a morphism $D_{nl,S} \to D_{nl',S}$ of atlases defined by $t \mapsto t^{l/l'}$. Thus we obtain a twisted jet $\mathcal{D}_{n,S}^{l'} \to \mathcal{X}$ over $\mathcal{X}$ and a morphism $\mathcal{J}_n\mathcal{Y} \to \mathcal{J}_n\mathcal{X}$. □

Lemma 2.15 (Canonical factorization)**.** *Let $f : \mathcal{W} \to \mathcal{V}$ be a morphism of DM stacks. Then there are a DM stack $\mathcal{W}'$ and morphisms $g : \mathcal{W} \to \mathcal{W}'$ and $h : \mathcal{W}' \to \mathcal{V}$ such that*

(1) $f = h \circ g$,
(2) *h is representable, and*

(3) *(universality) Let $g' : \mathcal{W} \to \mathcal{Z}$ and $h' : \mathcal{Z} \to \mathcal{V}$ be morphisms of DM stacks. If h' is representable and if the diagram of solid arrows*

$$\begin{tikzcd} \mathcal{W} \arrow[r, "g'"] \arrow[d, "g"'] & \mathcal{Z} \arrow[d, "h'"] \\ \mathcal{W}' \arrow[r, "h"'] \arrow[ur, dashed, "i"] & \mathcal{V} \end{tikzcd}$$

is commutative, then there is a morphism $i : \mathcal{W}' \to \mathcal{Z}$ making the whole diagram commutative, which is unique up to unique 2-isomorphism.

Proof. Let $M \to \mathcal{V}$ be an atlas. It induces a groupoid space $M \times_{\mathcal{V}} M \rightrightarrows M$. The stack associated to the groupoid space is canonically identified with $\mathcal{V}$. Let $\mathcal{W}_M := \mathcal{W} \times_{\mathcal{V}} M$. Similarly we can obtain some structure $\mathcal{W}_M \times_{\mathcal{W}} \mathcal{W}_M \rightrightarrows \mathcal{W}_m$ like a groupoid space, but each object is not a variety but a DM stack. Taking the coarse moduli space of the objects, we obtain a groupoid space $\overline{\mathcal{W}_M \times_{\mathcal{W}} \mathcal{W}_M} \rightrightarrows \overline{\mathcal{W}_M}$. Note that the existence of the coarse moduli space was proved by Keel and Mori [KM]. Let $N \to \mathcal{W}_M$ be an atlas. Then the composite $N \to \mathcal{W}_M \to \mathcal{W}$ is also an atlas. So we have another groupoid space $N \times_{\mathcal{W}} N \rightrightarrows N$. Now there are naturally induced morphisms of the groupoid spaces

$$(N \times_{\mathcal{W}} N \rightrightarrows N) \to (\overline{\mathcal{W}_M \times_{\mathcal{W}} \mathcal{W}_M} \rightrightarrows \overline{\mathcal{W}_M}) \to (M \times_{\mathcal{V}} M \rightrightarrows M),$$

and the corresponding morphisms of DM stacks

$$\mathcal{W} \to \mathcal{W}' \to \mathcal{V}$$

where $\mathcal{W}'$ is the stack associated to $\overline{\mathcal{W}_M \times_{\mathcal{W}} \mathcal{W}_M} \rightrightarrows \overline{\mathcal{W}_m}$. They clearly satisfy Conditions 1 and 2.

Suppose that there are morphisms $g' : \mathcal{W} \to \mathcal{Z}$ and $h' : \mathcal{Z} \to \mathcal{V}$ as in Condition 3. Taking the fiber products $\times_{\mathcal{V}} M$, we obtain the following diagram of solid arrows

$$\begin{tikzcd} \mathcal{W}_M \arrow[r] \arrow[d] & \mathcal{Z}_M \arrow[d] \\ \overline{\mathcal{W}_M} \arrow[r] \arrow[ur, dashed, "i_M"] & M. \end{tikzcd}$$

Since h' is representable, $\mathcal{Z}_M$ is isomorphic to an algebraic space. Now from the universality of the coarse moduli space, there is a morphism i_M making the whole diagram commutative, which is unique up to unique 2-isomorphism. It implies the last condition. □

Because of the universality, the morphisms g and h are uniquely determined. We call it the *canonical factorization* of f. For each point

$y \in \mathcal{Y}$ and its image $x \in \mathcal{X}$, we have a homomorphism of the automorphism groups $\phi : \mathrm{Aut}(y) \to \mathrm{Aut}(x)$. Then the canonical factorization corresponds to the factorization of ϕ,

$$\mathrm{Aut}(y) \twoheadrightarrow \mathrm{Im}(\phi) \hookrightarrow \mathrm{Aut}(x).$$

The canonical factorization is a generalization of the coarse moduli space. Indeed, in the lemma, if $\mathcal{V}$ is an algebraic space, then $\mathcal{W}'$ is the coarse moduli space of $\mathcal{W}$.

3. Motivic integration

3.1. **Convergent stacks.** In this subsection, we construct the semiring in which integrals take values.

Definition 3.1. A *convergent stack* is the pair $(\mathcal{X}, \alpha)$ of a DM stack $\mathcal{X}$ and a function

$$\alpha : \{\text{connected component of } \mathcal{X}\} \to \mathbb{Z}$$

such that

(1) there are at most countably many connected components,
(2) all connected components are of finite type, and
(3) for every $m \in \mathbb{Z}$, there are at most finitely many connected components $\mathcal{V}$ with $\dim \mathcal{V} + \alpha(\mathcal{V}) > m$.

We say that $\mathcal{X}$ is the *underlying stack* of $(\mathcal{X}, \alpha)$.

We abbreviate $(\mathcal{X}, \alpha)$ as $\mathcal{X}$, if it causes no confusion. A DM stack of finite type $\mathcal{X}$ is identified with the convergent stack $(\mathcal{X}, 0)$.

A *morphism* $f : (\mathcal{Y}, \beta) \to (\mathcal{X}, \alpha)$ of convergent stacks is a morphism $f : \mathcal{Y} \to \mathcal{X}$ of the underlying stacks with $\beta = \alpha \circ f$. A morphism of convergent stacks is called an *isomorphism* if it is an isomorphism of the underlying stacks. We say that convergent stacks $\mathcal{X}$ and $\mathcal{Y}$ are *isomorphic* (write $\mathcal{X} \cong \mathcal{Y}$) if there is an isomorphism between them.

If $(\mathcal{X}, \alpha)$ is a convergent stack and $\mathcal{X}'$ is a locally closed substack of $\mathcal{X}$, and if $\iota : \mathcal{X}' \hookrightarrow \mathcal{X}$ is the inclusion map, then $(\mathcal{X}', \alpha \circ \iota)$ is a convergent stack We call this a *convergent substack* of $\mathcal{X}$.

In the category of convergent stacks, we have the disjoint union and the product of two objects: Let $(\mathcal{X}, \alpha)$ and $(\mathcal{Y}, \beta)$ be convergent stacks. The disjoint union $(\mathcal{X}, \alpha) \sqcup (\mathcal{Y}, \beta)$ of them is the convergent stack $(\mathcal{X} \sqcup \mathcal{Y}, \gamma)$ such that $\gamma|_{\mathcal{X}} = \alpha$ and $\gamma|_{\mathcal{Y}} = \beta$. The product $(\mathcal{X}, \alpha) \times (\mathcal{Y}, \beta)$ is the convergent stack $(\mathcal{X} \times \mathcal{Y}, \gamma)$ such that for a connected component $\mathcal{V} \subset \mathcal{X} \times \mathcal{Y}$, $\gamma(\mathcal{V}) = \alpha(p_1(\mathcal{V})) + \beta(p_2(\mathcal{V}))$.

Definition 3.2. For a convergent stack $\mathcal{X} = (\mathcal{X}, \alpha)$, we define the *dimension* of $\mathcal{X}$, denoted $\dim \mathcal{X}$, to be

$$\max\{\dim \mathcal{V} + \alpha(\mathcal{V}) | \mathcal{V} \subset \mathcal{X} \text{ connected component}\}.$$

By convention, we put $\dim \emptyset = -\infty$.

Definition 3.3. Let $\mathfrak{R}'$ be the set of the isomorphism classes of convergent stacks. For each $n \in \mathbb{Z}$, we define $\sim_n$ to be the strongest equivalence relation of $\mathfrak{R}'$ satisfying the following basic relations:

(1) If $\mathcal{X}$ and $\mathcal{Y}$ are convergent stacks with $\dim \mathcal{Y} < n$, then $\mathcal{X} \sim_n \mathcal{X} \sqcup \mathcal{Y}$.
(2) If $\mathcal{X} \subset \mathcal{Y}$ is a convergent closed substack, then $\mathcal{Y} \sim_n (\mathcal{Y} \backslash \mathcal{X}) \sqcup \mathcal{X}$. (Here $\sqcup$ is not the disjoint union in $\mathcal{Y}$.)
(3) Let $(\mathcal{Y}, \beta)$ and $(\mathcal{X}, \alpha)$ be convergent stacks and $f : \mathcal{Y} \to \mathcal{X}$ a representable morphism of underlying stacks. If every geometric point y of $\mathcal{Y}$, $f^{-1}f(y)$ is isomorphic to an affine space of dimension $\alpha(f(y)) - \beta(y)$, then $(\mathcal{Y}, \beta) \sim_n (\mathcal{X}, \alpha)$.

Namely $\mathcal{X} \sim_n \mathcal{Y}$ if and only if $\mathcal{X}$ and $\mathcal{Y}$ can be connected by finitely many basic relations above. We define an equivalence relation $\sim$ of $\mathfrak{R}'$ as follows: For $a, b \in \mathfrak{R}'$, $a \sim b$ if and only if $a \sim_n b$ for all $n \in \mathbb{Z}$.

For example, we have

$$\begin{aligned}(\operatorname{Spec} k, 0) &\sim (\mathbb{A}^1_k, -1)\\ &\sim (\mathbb{A}^1_k \setminus \{0\}, -1) \sqcup (\mathbb{A}^1, -2)\\ &\sim \coprod_{i \in \mathbb{N}} (\mathbb{A}^1_k \setminus \{0\}, -i).\end{aligned}$$

Definition 3.4. We define $\mathfrak{R}$ to be $\mathfrak{R}'$ modulo $\sim$. For a convergent stack $\mathcal{X}$, we denote by $\{\mathcal{X}\}$ the equivalence class of $\mathcal{X}$.

Lemma 3.5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be convergent stacks. If $\mathcal{X} \sim \mathcal{Y}$, then* $\dim \mathcal{X} = \dim \mathcal{Y}$.

Proof. By definition, for all n, $\mathcal{X}$ and $\mathcal{Y}$ are connected by finitely many basic relations in Definition 3.3. For $n \ll 0$, these relations preserve the dimension. Hence $\dim \mathcal{X} = \dim \mathcal{Y}$. □

From the lemma, we have a map

$$\dim : \mathfrak{R} \to \mathbb{Z} \cup \{-\infty\}, \ \{\mathcal{X}\} \mapsto \dim \mathcal{X}.$$

Lemma 3.6. *Let $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be convergent stacks. If $\mathcal{X}_1 \sim \mathcal{X}_2$ and $\mathcal{Y}_1 \sim \mathcal{Y}_2$, then $\mathcal{X}_1 \sqcup \mathcal{Y}_1 \sim \mathcal{X}_2 \sqcup \mathcal{Y}_2$ and $\mathcal{X}_1 \times \mathcal{Y}_1 \sim \mathcal{X}_2 \times \mathcal{Y}_2$.*

Proof. The assertion $\mathcal{X}_1 \sqcup \mathcal{Y}_1 \sim \mathcal{X}_2 \sqcup \mathcal{Y}_2$ is clear.

If $\mathcal{X}_1 \sim_n \mathcal{X}_2$, then $\mathcal{X}_1$ and $\mathcal{X}_2$ are connected by the basic relations in Definition 3.3. Therefore $\mathcal{X}_1 \times \mathcal{Y} \sim_{n+\dim \mathcal{Y}} \mathcal{X}_2 \times \mathcal{Y}$ for any convergent stack $\mathcal{Y}$. This implies that if $\mathcal{X}_1 \sim \mathcal{X}_2$, then $\mathcal{X}_1 \times \mathcal{Y} \sim \mathcal{X}_2 \times \mathcal{Y}$. This proves the second assertion. □

We define a commutative semiring structure on $\mathfrak{R}$ by $\{\mathcal{X}\} + \{\mathcal{Y}\} := \{\mathcal{X} \sqcup \mathcal{Y}\}$ and $\{\mathcal{X}\}\{\mathcal{Y}\} := \{\mathcal{X} \times \mathcal{Y}\}$. The element $\{\emptyset\}$ is the unit of addition and the element $\{\operatorname{Spec} k\} = \{(\operatorname{Spec} k, 0)\}$ is the unit of product. We denote the element $\{\mathbb{A}^1_k\} = \{(\operatorname{Spec} k, 1)\}$ by $\mathbb{L}$. This is clearly invertible ($\mathbb{L}^{-1} = \{(\operatorname{Spec} k, -1)\}$).

Let I be a countable set and let $\mathcal{X}_i$, $i \in I$, be convergent stacks. If for every $m \in \mathbb{Z}$, there are at most finitely many $i \in I$ with $\dim \mathcal{X}_i > m$, then the disjoint union $\coprod_{i \in I} \mathcal{X}_i$ is also a convergent stack. In other words, if $x_i \in \mathfrak{R}$, $i \in I$ and if there are finitely many $i \in I$ with $\dim x_i > m$, then the infinite sum $\sum_{i \in I} x_i$ is defined. Note that $\sum_{i \in I} x_i$ is independent of the order of I. Moreover the following holds: If $I_j \subset I$ are subsets with $I = \coprod_j I_j$, then we have

$$\sum_{i \in I} x_i = \sum_j \sum_{i \in I_j} x_i.$$

Definition 3.7. We define $\bar{\mathfrak{R}} := \mathfrak{R} \cup \{\infty\}$.

This is still a semigroup by $x + \infty = \infty$ for any $x \in \bar{\mathfrak{R}}$. Let I be a countable set and $x_i \in \bar{\mathfrak{R}}$, $i \in I$. Either if there exists $i \in I$ with $x_i = \infty$ or if for some $m \in \mathbb{Z}$, there exist infinitely many i with $\dim x_i > m$, then we define $\sum_{i \in I} x_i := \infty$. Thus in the semigroup $\bar{\mathfrak{R}}$, the sum of arbitrary countable collection of elements is defined.

We can see the following basic properties of the map dim.

Lemma 3.8. (1) *For* $x, y \in \mathfrak{R}$, $\dim(x \times y) = \dim x + \dim y$.
(2) *Let* I *be a countable set and* $x_i \in \mathfrak{R}$, $i \in I$. *If* $\sum_{i \in I} x_i \neq \infty$, *then* $\dim \sum_{i \in I} x_i = \max\{\dim x_i | i \in I\}$.

Remark 3.9. Originally, the motivic integration was defined in a ring deriving from the Grothendieck ring of varieties, in precise, a completion of a localization of the Grothendieck ring (see [DL1]). In the Grothendieck ring, there are negative elements, that is, there is the inverse of any class of varieties. This causes some problems. For example, the Grothendieck ring of varieties is not a domain [Poo]. However we do not really need negative elements to construct the integration theory. Thus it is natural to replace the Grothendieck ring with the Grothendieck *semiring*.

However we do not superficially use the Grothendieck semiring. Instead, we consider a sufficiently large set (the set of the isomorphism classes of convergent stacks or spaces) and take its quotient by relations needed below. Advantages of this approach are that each element of the obtained semiring corresponds to a "geometric" object and that we do not have to use algebraic operations on semirings like the localization and the completion.

3.2. **Convergent spaces and coarse moduli spaces.** A *convergent space* is defined to be a convergent stack $(\mathcal{X}, \alpha)$ such that the underlying stack $\mathcal{X}$ is an algebraic space.

Definition 3.10. Let $\mathfrak{S}'$ be the set of the isomorphism classes of convergent spaces. For each $n \in \mathbb{Z}$, we define $\simeq_n$ to be the strongest equivalence relation of $\mathfrak{S}'$ satisfying the following basic relations:

(1) If $f : Y \to X$ is a morphism of convergent spaces which is finite, surjective and universally injective, then $X \simeq_n Y$.
(2) If X and Y are convergent spaces with $\dim Y < n$, then $X \simeq_n X \sqcup Y$.
(3) If $X \subset Y$ is a convergent closed subspace, then $Y \simeq_n (Y \setminus X) \sqcup X$.
(4) Let (Y, β) and (X, α) be convergent spaces and $f : Y \to X$ a morphism of underlying algebraic spaces. If every geometric point $y \in Y(K)$, there is a finite, surjective and universally injective morphism
$$\mathbb{A}_K^{\alpha(f(y))-\beta(y)}/G \to f^{-1}f(y)$$
for some finite group action $G \curvearrowright \mathbb{A}_K^{\alpha(f(y))-\beta(y)}$, then $(Y, \beta) \simeq_n (X, \alpha)$.

We define an equivalence relation $\simeq$ of $\mathfrak{R}'$ as follows: For $a, b \in \mathfrak{R}'$, $a \simeq b$ if and only if $a \simeq_n b$ for all $n \in \mathbb{Z}$.

A finite, surjective and universally injective morphism induces an equivalence of étale sites [SGA1, IX. Théorème 4.10] and an isomorphism of étale cohomology. We will use this fact in the following subsection.

Definition 3.11. We define $\mathfrak{S}$ to be $\mathfrak{S}'$ modulo $\simeq$.

The set $\mathfrak{S}$ has a semiring structure as $\mathfrak{R}$ does.

For a DM stack $\mathcal{X}$, we denote by $\bar{\mathcal{X}}$ its coarse moduli space. For a convergent stack $\mathcal{X} = (\mathcal{X}, \alpha)$, we can give a natural convergent space structure to the coarse moduli space $\bar{\mathcal{X}}$. We denote this convergent space also by $\bar{\mathcal{X}}$.

Proposition 3.12. *There is a semiring homomorphism*

$$\mathfrak{R} \to \mathfrak{S},\ \{\mathcal{X}\} \mapsto \{\bar{\mathcal{X}}\}.$$

Proof. First, to show that there is a map of sets, we have to show that for each n, if $\mathcal{X}$ and $\mathcal{Y}$ are in one of the basic relations in Definition 3.3, then $\bar{\mathcal{X}} \simeq_n \bar{\mathcal{Y}}$.

Concerning the first basic relation, this is clear. We next consider the second one. If $\mathcal{X} \subset \mathcal{Y}$ is a convergent closed substack and $X \subset \bar{\mathcal{Y}}$ is a convergent closed subspace which is the image of $\mathcal{Y}$, then the natural morphism $\bar{\mathcal{X}} \to X$ is finite, surjective and universally injective. Thus $\bar{\mathcal{Y}} \simeq_n \overline{\mathcal{Y} \setminus \mathcal{X}} \sqcup \bar{\mathcal{Y}}$.

Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of DM stacks and $g : Y \to X$ the corresponding morphism of the coarse moduli spaces. Let $x : \operatorname{Spec} K \to \mathcal{X}$ be a geometric point and $\bar{x} : \operatorname{Spec} K \to X$ the corresponding point of the coarse moduli space. Suppose that $f^{-1}(x) \cong \mathbb{A}^d_K$. Then there is a natural morphism $\mathbb{A}^d_K/\mathrm{Aut}(x) \to g^{-1}(\bar{x})$ which is finite, surjective and universally injective. This implies that if $\mathcal{X}$ and $\mathcal{Y}$ is in the last basic relation in Definition 3.3, then $\bar{\mathcal{X}}$ and $\bar{\mathcal{Y}}$ are in the last basic relation in Definition 3.10.

Thus we have a map $\mathfrak{R} \to \mathfrak{S},\ \{\mathcal{X}\} \mapsto \{\bar{\mathcal{X}}\}$.

Next, to show that this map is a semiring homomorphism, we have to show that

$$\{\bar{\mathcal{X}}\}\{\bar{\mathcal{Y}}\} := \{\bar{\mathcal{X}} \times \bar{\mathcal{Y}}\} = \{\overline{\mathcal{X} \times \mathcal{Y}}\}.$$

There is a natural morphism $\overline{\mathcal{X} \times \mathcal{Y}} \to \bar{\mathcal{X}} \times \bar{\mathcal{Y}}$. This is finite, surjective and universally injective. It follows that $\{\bar{\mathcal{X}} \times \bar{\mathcal{Y}}\} = \{\overline{\mathcal{X} \times \mathcal{Y}}\}$. $\square$

3.3. **Cohomology realization.** To control huge semirings $\mathfrak{R}$ and $\mathfrak{S}$, it is useful to consider homomorphisms from these semirings to smaller and easier ones. The cohomology theory helps to construct such homomorphisms.

3.3.1. *The Grothendieck ring of an abelian category.* Let $\mathcal{C}$ be an abelian category. Its *Grothendieck ring* $K_0(\mathcal{C})$ is defined to be the free abelian group generated by isomorphism classes $[M]$ of objects M of $\mathcal{C}$ modulo the following relation: If there is a short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

then $[M_2] = [M_1] + [M_3]$. It is easy to see that given a sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$, then we have an equation

$$[M] = \sum_{i=0}^{n-1} [M_{i+1}/M_i].$$

Every object M of $\mathcal{C}$ has a Jordan-Hölder sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$, that is, all M_{i+1}/M_i are simple objects. Then the *semisimplification* M^{ss} is by definition the associated graded $\bigoplus_i M_{i+1}/M_i$ of the sequence. It is a semisimple object and its isomorphism class depends only on M. We have $[M] = [M^{ss}]$. If M_1 and M_2 are semisimple and $[M_1] = [M_2] \in K_0(\mathcal{C})$, then M_1 and M_2 are isomorphic. If M_1 and M_2 are arbitrary objects and $[M_1] = [M_2] \in K_0(\mathcal{C})$, then $M_1^{ss} \cong M_2^{ss}$.

3.3.2. *Mixed Hodge structures.* Now we consider the case where $\mathcal{C}$ is the category of (rational) mixed Hodge structures, denoted MHS. Since every mixed Hodge structure has, by definition, a weight filtration whose associated graded is a pure Hodge structure, the Grothendieck ring $K_0(MHS)$ is, in fact, generated by the classes of pure Hodge structures. For a variety X over $\mathbb{C}$, we define

$$\chi_h(X) := \sum_i (-1)^i [H_c^i(X, \mathbb{Q})] \in K_0(MHS).$$

For a variety X and its closed subvariety V, since there is the localization sequence

$$\cdots \to H_c^i(X \setminus V) \to H_c^i(X) \to H_c^i(V) \to H_c^{i+1}(X \setminus V) \to \cdots,$$

we have $\chi_h(X) = \chi_h(X \setminus V) + \chi_h(V)$. We denote $[\mathbb{Q}(-1)] = \chi_h(\mathbb{A}_k^1)$ by $\mathbb{L}$.

3.3.3. *p-adic Galois representations.* Suppose that k is a finite field. Let p be a prime number different from the characteristic of k and $\mathbb{Q}_p$ be the p-adic field. Then the compact-supported p-adic cohomology groups $H_c^i(X \otimes \bar{k}, \mathbb{Q}_p)$ are finite dimensional $\mathbb{Q}_p$-vector spaces with continuous actions of the absolute Galois group $\mathbf{G}_k = \mathrm{Gal}(\bar{k}/k)$.

Let V be an arbitrary Galois representation, that is, a finite dimensional $\mathbb{Q}_p$-vector space with continuous $\mathbf{G}_k$-action. We say that V is *pure of weight* $i \in \mathbb{Q}$ if every eigenvalue $\alpha \in \bar{\mathbb{Q}}_p$ of the Frobenius action on V is algebraic and all complex conjugates of α have absolute value $p^{-i/2}$. We say that V is *mixed (of weight* $\le i_n, \ge i_0$*)* if there is a filtration $0 = W_{i_0} \subset W_{i_1} \subset \cdots \subset W_{i_n} = V$, $(i_j \in \mathbb{Q})$ such that $W_{i_j}/W_{i_{j-1}}$ is pure of weight i_j. (We admit rational weights for later use.)

Let $MR(\mathbf{G}_k, \mathbb{Q}_p)$ be the category of mixed Galois representations. This category is abelian. For a variety X, $H_c^i(X \otimes_k \bar{k}, \mathbb{Q}_p)$ is mixed [Del2]. Define

$$\chi_p(X) := \sum_i (-1)^i [H_c^i(X \otimes_k \bar{k}, \mathbb{Q}_p)] \in K_0(MR(\mathbf{G}_k, \mathbb{Q}_p)).$$

Then this invariant have the same properties as χ_h, namely, $\chi_p(X) = \chi_p(X \setminus V) + \chi_p(V)$ for a closed subvariety $V \subset X$, and $\chi_p(X \times Y) = \chi_p(X)\chi_p(Y)$.

3.3.4. *Completions of Grothendieck rings.* Let $F_m K_0(MHS) \subset K_0(MHS)$ be the subgroup generated by the elements $[H]$ with H of weight $\le -2m$. We define the completion

$$\hat{K}_0(MHS) := \varprojlim K_0(MHS)/F_m K_0(MHS).$$

Since $F_m K_0(MHS) \cdot F_m K_0(MHS) \subset F_{m+n} K_0(MHS)$, $\hat{K}_0(MHS)$ has a ring structure.

Lemma 3.13. *The natural map $K_0(MHS) \to \hat{K}_0(MHS)$ is injective.*

Proof. Let $\alpha = \sum_i n_i[H_i] \in K_0(MHS)$. Then if we put $\alpha_w := \sum n_i[\mathrm{Gr}_W^w H_i]$, we have that $\alpha = \sum_w \alpha_w$ and that $\alpha = 0$ if and only if $\alpha_w = 0$ for every w. Moreover $\alpha \in F_m$ if and only if $\alpha_w = 0$ for every $w > -2m$. Now we can see that $\bigcap_m F_m K_0(MHS) = \{0\}$ and the completion map is injective. $\square$

Next we define a completion of $K_0(MR(\mathbf{G}_k, \mathbb{Q}_p))$. For each integer m, we define $F_m K_0(MR(\mathbf{G}_k, \mathbb{Q}_p)) \subset K_0(MR(\mathbf{G}_k, \mathbb{Q}_p))$ to be the subgroup generated by the elements $[V]$ with V of weight $\le -2m$. We define

$$\hat{K}_0(MR(\mathbf{G}_k, \mathbb{Q}_p)) := \varprojlim K_0(MR(\mathbf{G}_k, \mathbb{Q}_p))/F_m K_0(MR(\mathbf{G}_k, \mathbb{Q}_p)).$$

This is also a ring. The natural map $K_0(MR(\mathbf{G}_k, \mathbb{Q}_p)) \to \hat{K}_0(MR(\mathbf{G}_k, \mathbb{Q}_p))$ is also injective.

3.3.5. *Maps from $\mathfrak{R}$ and $\mathfrak{S}$.* When $k = \mathbb{C}$, for a convergent space $X = (X, \alpha)$, we define

$$\chi_h(X) := \sum_{V \subset X} \chi_h(V)\mathbb{L}^{\alpha(V)} \in \hat{K}_0(MHS).$$

Here the sum runs over all connected components V of X. By definition, for every $m \in \mathbb{Z}$, there are at most finitely many V with $\dim V + \alpha(V) > m$. Since $\chi_h(V)$ is of weight $\le 2\dim V$, for every $m \in \mathbb{Z}$, there are at most finitely many V with $\chi_h(V)\mathbb{L}^{\alpha(V)} \notin F_{-m}K_0(MHS)$. Therefore $\chi_h(X)$ is well-defined.

When k is a finite field, for a convergent space X, we similarly define

$$\chi_p(X) := \sum_{V \subset X} \chi_p(V)\mathbb{L}^{\alpha(V)} \in \hat{K}_0(MR(\mathbf{G}_k, \mathbb{Q}_p)).$$

Proposition 3.14. *When $k = \mathbb{C}$, we have a semiring homomorphism*

$$\mathfrak{S} \to \hat{K}_0(MHS),\ \{X\} \mapsto \chi_h(X).$$

When k is a finite field, we have a semiring homomorphism

$$\mathfrak{S} \to \hat{K}_0(MR(\mathbf{G}_k, \mathbb{Q}_p)),\ \{X\} \mapsto \chi_p(X).$$

Proof. If the maps are well-defined, then these are obviously semiring homomorphisms. Now it suffices to show that for every $n \in \mathbb{Z}$, the basic relations in Definition 3.10 preserves $\chi_h(X)$ modulo $F_{-n}\hat{K}_0(MHS)$ or $\chi_p(X)$ modulo $F_{-n}\hat{K}_0(MR(\mathbf{G}_k, \mathbb{Q}_p))$. Here $F_{-n}\hat{K}_0(MHS)$ and $F_{-n}\hat{K}_0(MR(\mathbf{G}_k, \mathbb{Q}_p))$ are the completions of $F_{-n}K_0(MHS)$ and $F_{-n}K_0(MR(\mathbf{G}_k, \mathbb{Q}_p))$. The second basic relation clearly does.

The basic relations except for the second one, in fact, preserves $\chi_h(X)$ and $\chi_p(X)$ even in $\hat{K}_0(MHS)$ and $\hat{K}_0(MR(\mathbf{G}_k, \mathbb{Q}_p))$. Only the fourth one is not trivial. This results from Lemma 3.15 and the fact that a finite, surjective and universally injective morphism is an homeomorphism both in analytic topology and in étale topology (for étale topology, see [SGA1, IX. Théorème 4.10]). □

Lemma 3.15. *Suppose that k is either $\mathbb{C}$ or a finite field. Let $f : Y \to X$ be a morphism of varieties and d a positive integer. Suppose that for every geometric point $x \in X(K)$, the fiber $f^{-1}(x)$ has the same compact-supported cohomology as $\mathbb{A}^d_K$. Then we have*

$$\chi_h(Y) = \chi_h(X)\mathbb{L}^d \ (k = \mathbb{C})$$
$$\chi_p(Y) = \chi_p(X)\mathbb{L}^d \ (k \text{ finite field}).$$

Proof. We discuss χ_h and χ_p together and write $\mathbb{Q}_p$ as $\mathbb{Q}$. There is a spectral sequence

$$E_2^{i,j} = H_c^i(X, R^j f_! \mathbb{Q}) \Rightarrow H_c^{i+j}(Y, \mathbb{Q})$$

From Grothendieck's generic flatness, we may assume that f is flat. Then from the assumption, we have

$$R^j f_! \mathbb{Q} = \begin{cases} \mathbb{Q}(-d) & (j = 2d) \\ 0 & (j \neq 2d), \end{cases}$$

and the spectral sequence degenerates. Hence we have an isomorphism

$$H_c^{i+2d}(Y, \mathbb{Q}) \cong H_c^i(X, \mathbb{Q}) \otimes \mathbb{Q}(-d),$$

which implies the lemma. □

Composing maps, we obtain semiring homomorphisms

$$\mathfrak{R} \to \hat{K}_0(MHS),\ \mathcal{X} \mapsto \chi_h(\bar{\mathcal{X}}),\ \text{and}$$
$$\mathfrak{R} \to \hat{K}_0(MR(\mathbf{G}_k, \mathbb{Q}_p)),\ \mathcal{X} \mapsto \chi_p(\bar{\mathcal{X}}).$$

3.4. **Cylinders and motivic measure.** Let $\mathcal{X}$ be a smooth DM stack $\mathcal{X}$ of finite type and pure dimension d.

Definition 3.16. Let $n \in \mathbb{Z}_{\geq 0}$. A subset $A \subset |\mathcal{J}_\infty \mathcal{X}|$ is said to be an *n-cylinder* if $A = \pi_n^{-1}\pi_n(A)$ and $\pi_n(A)$ is a constructible subset. A subset $A \subset |\mathcal{J}_\infty \mathcal{X}|$ is said to be a *cylinder* if it is an n-cylinder for some n.

The collection of cylinders is stable under finite unions and finite intersections. For a cylinder A and $n \in \mathbb{Z}_{\geq 0}$ such that A is an n-cylinder, we define

$$\mu_{\mathcal{X}}(A) := \{\pi_n(A)\}\mathbb{L}^{-nd} \in \mathfrak{R}.$$

This is independent of n, thanks to Lemma 3.18.

Remark 3.17. In some articles, $\tilde{\mu}_{\mathcal{X}}(A)$ is defined to be $\{\pi_n(A)\}\mathbb{L}^{-(n+1)d}$ instead of $\{\pi_n(A)\}\mathbb{L}^{-nd}$. However the difference is just superficial.

Lemma 3.18. *Let $\mathcal{X}$ be a smooth DM stack of pure dimension d. Let $p : \mathcal{J}_{n+1}\mathcal{X} \to \mathcal{J}_n\mathcal{X}$ be the natural projection. Then for every geometric point $z \in (\mathcal{J}_n\mathcal{X})(K)$, the fiber $p^{-1}(z)$ is isomorphic to $\mathbb{A}^d_K$.*

Proof. We may assume that $\mathcal{X}$ is a quotient stack $[M/G]$. Then $\mathcal{J}^l_n\mathcal{X} \cong \coprod_{a \in \mathrm{Conj}^l(G)}[J^{(a)}_{nl}M/C_a]$. Therefore it suffices to show that the fiber of the morphism $J^{(a)}_{(n+1)l}M \to J^{(a)}_{nl}M$ over any geometric point is isomorphic to an affine space of dimension d. Now we can take a completion of M at a K-point w, $\hat{M} = \operatorname{Spec} K[[x_1, \dots, x_d]]$. Let $a : \boldsymbol{\mu}_l \hookrightarrow G$ be an embedding. By the natural morphism $J^{(a)}_{nl}\hat{M} \to J^{(a)}_{nl}M$, the space $(J^{(a)}_{nl}\hat{M})(K)$ is identified with the fiber of $J^{(a)}_{nl}M \to M$ over w. Assume that the $\boldsymbol{\mu}_l$-action on M through a is diagonal and that $\zeta \in \boldsymbol{\mu}_l$ sends x_i to $\zeta^{a_i}x_i$ with $1 \leq a_i \leq l$. Then $(J^{(a)}_{nl}M)(K)$ parameterizes the homomorphisms $K[[x_1, \dots, x_d]] \to K[[t]]/t^{nl+1}$ which send x_i to an element of the form

$$c_0t^{a_i} + c_1t^{l+a_i} + c_2t^{2l+a_i} \cdots + c_{n-1}t^{(n-1)l+a_i},\ c_i \in K.$$

Therefore the fiber of $J^{(a)}_{(n+1)l}M \to J^{(a)}_{nl}M$ over every geometric point is isomorphic to an affine space of dimension d. $\square$

It is obvious that $\mu_{\mathcal{X}}$ is a finite additive measure:

Proposition 3.19. *If A and A_i $(i = 1, \dots, n)$ are cylinders such that $A = \coprod_{1 \le i \le n} A_i$, then*

$$\mu_{\mathcal{X}}(A) = \sum_{i=1}^{n} \mu_{\mathcal{X}}(A_i).$$

3.5. **Integrals of measurable functions.**

Definition 3.20. Let $A \subset |\mathcal{J}_\infty \mathcal{X}|$ be a subset. A function $F : A \to \mathfrak{R}$ is said to be *measurable* if there are countably many cylinders A_i such that $A = \coprod_i A_i$ and the restriction of F to each A_i is constant.

We define the integral of a measurable function $F : A \to \mathfrak{R}$ as follows;

$$\int_A F d\mu_{\mathcal{X}} := \sum F(A_i) \cdot \mu_{\mathcal{X}}(A_i) \in \bar{\mathfrak{R}}.$$

Definition 3.21. Let $A \subset |\mathcal{J}_\infty \mathcal{X}|$ be a cylinder. If $n \in \mathbb{Z}_{\ge 0}$ is such that A is an n-cylinder, we define

$$\operatorname{codim} A := \operatorname{codim} (\pi_n(A), |\mathcal{J}_n \mathcal{X}|).$$

Lemma 3.22. *The integral of a measurable function F is independent of the choice of A_i.*

Proof. Let A_i, B_i, $i \in \mathbb{N}$ be cylinders such that $F|_{A_i}$ and $F|_{B_i}$ are constant for all i and $A = \coprod_{i \in \mathbb{N}} A_i = \coprod_{i \in \mathbb{N}} B_i$. We assume that $\{B_i\}$ is a refinement of $\{A_i\}$. The general case can be reduced to this case. Let $S_i := \{j \in \mathbb{N} | B_j \subset A_i\}$, let $\eta_1, \dots, \eta_u$ be the generic points of the irreducible components of A_i and let $j_v \in S_i$, $1 \le v \le u$ be such that $\eta_v \in B_{j_v}$. Then we see that for $j' \in S_i \setminus \{j_1, \dots, j_u\}$,

$$\operatorname{codim} B_{j'} \ge \operatorname{codim} (A_i \setminus \bigcup_{1 \le v \le u} B_{j_v}) > \operatorname{codim} A_i.$$

Repeating this argument, we see that for every $m \in \mathbb{Z}$, there are at most finitely many $j \in S_i$ such that $\operatorname{codim} B_j < m$. It means that for every $n \in \mathbb{Z}$, there is a finite subset $S_{i,n} \subset S_i$ such that for $j \in S_i \setminus S_{i,n}$, $\dim \mu_{\mathcal{X}}(B_j) < n$. For every n, we have

$$\mu_{\mathcal{X}}(A_i) \sim_n \mu_{\mathcal{X}}(\coprod_{j \in S_{i,n}} B_j) \sim_n \sum_{j \in S_i} \mu_{\mathcal{X}}(B_j).$$

Hence $\mu_{\mathcal{X}}(A_i) = \sum_{j \in S_i} \mu_{\mathcal{X}}(B_j)$. This proves the lemma. (In fact, S_i is a finite set. See [Loo, Lemma 2.3]) $\square$

Below, we consider such functions as $\mathbb{L}^{\operatorname{ord} \mathcal{I}_{\mathcal{Z}}}$. Here $\operatorname{ord} \mathcal{I}_{\mathcal{Z}}$ is the order function associated to the ideal sheaf $\mathcal{I}_{\mathcal{Z}}$ of a closed substack $\mathcal{Z} \subset \mathcal{X}$

and takes values in $\mathbb{Z}_{\geq 0}$ outside $|\mathcal{J}_\infty \mathcal{Z}| \subset |\mathcal{J}_\infty \mathcal{X}|$ and the infinity on $|\mathcal{J}_\infty \mathcal{Z}|$. Thus a function $\mathbb{L}^{\mathrm{ord}\,\mathcal{I}_\mathcal{Z}}$ is not defined on $|\mathcal{J}_\infty \mathcal{Z}|$ (at least, as a $\mathfrak{R}$-valued function). However, if $\mathcal{Z} \subsetneqq \mathcal{X}$, then this does not cause any problem, because $|\mathcal{J}_\infty \mathcal{Z}|$ is too small to affect integrals.

Definition 3.23. A subset $A \subset |\mathcal{J}_\infty \mathcal{X}|$ is said to be *negligible* if there are cylinders A_i, $i \in \mathbb{N}$ such that $A = \bigcap_i A_i$ and $\lim_{i\to\infty} \operatorname{codim} A_i = \infty$.

Proposition 3.24. *Let $\mathcal{Z} \subset \mathcal{X}$ be a locally closed substack of dimension $d' < d$. Then a subset $|\mathcal{J}_\infty \mathcal{Z}| \subset |\mathcal{J}_\infty \mathcal{X}|$ is negligible.*

Proof. We have
$$|\mathcal{J}_\infty \mathcal{Z}| = \bigcap_{n\geq 0} \pi_n^{-1}\pi_n(|\mathcal{J}_\infty \mathcal{Z}|).$$
From Lemma 3.27, $\pi_n^{-1}\pi_n(|\mathcal{J}_\infty \mathcal{Z}|)$ are cylinders, and from Lemma 3.29,
$$\lim_{n\to\infty} \operatorname{codim} \pi_n^{-1}\pi_n(|\mathcal{J}_\infty \mathcal{Z}|) = \infty.$$
Hence $|\mathcal{J}_\infty \mathcal{Z}|$ is negligible. □

Removing a negligible subset from the domain or adding one to the domain does not change the value of integrals:

Proposition 3.25. *Let $F : A \to \mathfrak{R}$ be a function and $B \subset A$ a negligible subset. If F is measurable, then $F|_{A\setminus B}$ is measurable and*
$$\int_A F d\mu_\mathcal{X} = \int_{A\setminus B} F d\mu_\mathcal{X}.$$

Proof. Let A_i, $i \in \mathbb{N}$ be cylinders such that $A = \coprod_i A_i$ and $F|_{A_i}$ are constant for all i. Let B_i, $i \in \mathbb{N}$ be cylinders such that $B = \bigcap_i B_i$ and $\lim_{i\to\infty} \operatorname{codim} B_i \to \infty$. We may assume that $B_1 = A$ and $B_{i+1} \subset B_i$ for all i. We put $C_i := B_i \setminus B_{i+1}$. Then C_i are mutually disjoint cylinders such that $\coprod_{i\in\mathbb{N}} C_i = A \setminus B$ and $\lim_{i\to\infty} \operatorname{codim} C_i = \infty$. For $i, j \in \mathbb{N}$, we put $A_{ij} := A_i \cap C_j$. These are cylinders with $\coprod_{i,j} A_{ij} = A \setminus B$. Since $F|_{A_{ij}}$ are constant for all i, j, $F|_{A\setminus B}$ is measurable.

We have
$$\int_{A\setminus B} F d\mu_\mathcal{X} = \sum_i \sum_j F(A_{ij}) \cdot \mu_\mathcal{X}(A_{ij}).$$
Therefore, to prove the equation of the proposition, it suffices to show $\mu_\mathcal{X}(A_i) = \sum_j \mu_\mathcal{X}(A_{ij})$. By definition, we have $A_i \setminus B_{j_0+1} = \coprod_{j\leq j_0} A_{ij}$. For every $n \in \mathbb{Z}$, if j_0 is sufficiently large, then
$$\mu_\mathcal{X}(A_i) \sim_n \mu_\mathcal{X}(A_i \setminus B_{j_0+1}) = \mu_\mathcal{X}(\coprod_{j\leq j_0} A_{ij}) \sim_n \sum_j \mu_\mathcal{X}(A_{ij}).$$
Hence $\mu_\mathcal{X}(A_i) = \sum_j \mu_\mathcal{X}(A_{ij})$. □

By abuse of terminology, we say that F is a measurable function on a subset A even if F is a measurable function defined only on $A \setminus B$ with B negligible, and write

$$\int_A F d\mu_{\mathcal{X}} = \int_{A \setminus B} F d\mu_{\mathcal{X}}.$$

Remark 3.26. The definition of the measurable function in this paper differs from that in [DL2, Appendix].

3.5.1. *Lemmas.*

Lemma 3.27. *Let $\mathcal{Z}$ be a DM stack of finite type. Then there is a monotone increasing function $\phi : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that $\phi(n) \geq n$ for all n and*

$$\pi_n(|\mathcal{J}_\infty \mathcal{Z}|) = \mathrm{Im}(|\mathcal{J}_{\phi(n)} \mathcal{Z}| \to |\mathcal{J}_n \mathcal{Z}|).$$

In particular, $\pi_n(|\mathcal{J}_\infty \mathcal{Z}|) \subset |\mathcal{J}_n \mathcal{Z}|$ is a constructible subset.

In the case where $\mathcal{Z}$ is a scheme, this was proved by Greenberg [Gre] by using Newton-Hensel lemma (called Newton's lemma in [Gre]). We prove Lemma 3.27 by using the equivariant version of Newton-Hensel lemma (Lemma 3.28).

Proof. The second assertion of the lemma follows from the first and Chevalley's theorem for stacks [LMB, Théorème 5.9.4].

We may assume that $\mathcal{Z} \cong [Z/G]$ with $Z = \operatorname{Spec} R$ an affine scheme and G a finite group. Furthermore we may assume that k contains all l-th roots of unity for l prime to the characteristic of k such that there is an element $g \in G$ of order l. From Proposition 2.8, the first assertion of the lemma is equivalent to the following:

★ For every $a : \boldsymbol{\mu}_l \hookrightarrow G$ and for every $0 \leq n < \infty$, there is a monotone increasing function $\phi : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that for every n, $\phi(n) \geq n$ and

$$\mathrm{Im}(J^{(a)}_{\phi(n)l} Z \to J^{(a)}_{nl} Z) = \mathrm{Im}(J^{(a)}_\infty Z \to J^{(a)}_{nl} Z).$$

We prove ★ by the induction on $\dim Z$. There exists a $\boldsymbol{\mu}_l$-equivariant embedding of Z into $\mathbb{A}^d = \operatorname{Spec} k[x_1, \dots, x_d]$ on which $\boldsymbol{\mu}_l$ acts diagonally. Let $I \subset k[x_1, \dots, x_d]$ be the defining ideal of Z. For some positive integer s, we have $\sqrt{I} \supset I \supset (\sqrt{I})^s$. Let Z_{red} and Z^s_{red} be the closed subschemes defined by $\sqrt{I}$ and $(\sqrt{I})^s$ respectively. Then for every m, we have

$$J^{(a)}_{ml} Z_{\mathrm{red}} \subset J^{(a)}_{ml} Z \subset J^{(a)}_{ml} Z^s_{\mathrm{red}} \subset J^{(a)}_{ml} \mathbb{A}^d$$

and

$$J^{(a)}_\infty Z_{\mathrm{red}} = J^{(a)}_\infty Z = J^{(a)}_\infty Z^s_{\mathrm{red}}.$$

Hence it suffices to show ★ for Z_{red} and Z^s_{red}. If m' is such that $m'l+1 \geq mls+s$, then we have $\mathrm{Im}(J^{(a)}_{m'l}Z^s_{\mathrm{red}} \to J^{(a)}_{ml}\mathbb{A}^d) \subset J^{(a)}_{ml}Z_{\mathrm{red}}$. Hence it suffices to show ★ only for Z reduced.

Then ★ clearly holds if Z is of dimension zero.

We can also assume that Z is irreducible: Let $Z_1,\dots,Z_q$ be the irreducible components of Z and let $W := \bigcup_{i\neq j} Z_i \cap Z_j$. Since $\dim W < \dim Z$, by the inductive assumption, ★ holds for W. Every $\gamma \in J_\infty Z \setminus J_\infty W$ lies in only one component Z_i. Thus Assertion ★ for Z follows from ★ for $Z_1,\dots,Z_q$.

Suppose that Z is irreducible, reduced and of positive dimension. Let $f_1,\dots,f_s \in k[x_1,\dots,x_d]^{\boldsymbol{\mu}_l}$ be $\boldsymbol{\mu}_l$-invariant polynomials defining Z and $r := \mathrm{codim}\,(Z,\mathbb{A}^d) \leq s$. Reordering suitably $f_1,\dots,f_s$, we have that Z is an irreducible component of $V := V(f_1,\dots,f_r)$. Let

$$J := \det\left(\frac{\partial f_i}{\partial x_j}\right)_{1\leq i\leq r,\ 1\leq j\leq r}.$$

Since k is perfect and Z is reduced, Z is generically smooth. Therefore, if necessary, reordering variables $x_1,\dots,x_d$, we may assume that the subscheme $S := V(f_1,\dots,f_s,J) \subset Z$ is of dimension $< \dim Z$: Let W be the intersection of Z and the closure of $V \setminus Z$ and let $X \subset Z$ be the scheme-theoretic union of W and S, which is of dimension $< \dim Z$. By the inductive assumption, there exists a monotone increasing function ψ such that

$$\mathrm{Im}(J^{(a)}_\infty X \to J^{(a)}_{nl}X) = \mathrm{Im}(J^{(a)}_{\psi(n)l}X \to J^{(a)}_{nl}X).$$

Let $K \supset k$ be a field extension. A K-point γ of $J^{(a)}_\infty \mathbb{A}^d$ corresponds to a $\boldsymbol{\mu}_l$-equivariant $K[[t]]$-algebra homomorphism

$$\gamma^* : K[[t]][x_1,\dots,x_d] \to K[[t]].$$

Suppose that $\zeta \in \boldsymbol{\mu}_l$ sends x_i to $\zeta^{a_i}x_i$, $0 \leq a_i \leq l-1$. Let $\gamma \in (J^{(a)}_\infty \mathbb{A}^d)(K)$ be such that $\pi_{nl}(\gamma) \in J^{(a)}_{nl}Z \setminus J^{(a)}_{nl}X$ and let $nl \leq e < (n+1)l$ be the unique integer such that $e + \sum_{i=1}^r a_i \equiv 0 \mod l$. Since $\pi_{nl}(\gamma) \notin J^{(a)}_{nl}S$, we have

$$\gamma^*(J) \not\equiv 0 \mod (t^{nl+1}) \text{ (and } \mod (t^{e+1})).$$

From Lemma 3.28, if $\pi_{ml}(\gamma) \in J^{(a)}_{ml}V$ with $ml+1 \geq 2(e+\sum_{i=1}^r a_i)+l$, then there is $\gamma' \in J^{(a)}_\infty V$ such that $\pi_{nl}(\gamma') = \pi_{nl}(\gamma)$. Moreover $\pi_{ml}(\gamma)$ must lie in $J^{(a)}_{ml}Z$ and γ' in $J^{(a)}_\infty Z$. Hence there exists a monotone increasing function τ such that

$$\mathrm{Im}(J^{(a)}_\infty Z \to J^{(a)}_{nl}Z) \setminus J^{(a)}_{nl}X = \mathrm{Im}(J^{(a)}_{\tau(n)l}Z \to J^{(a)}_{nl}Z) \setminus J^{(a)}_{nl}X.$$

We show that ★ holds for $\phi := \tau \circ \psi$. Let $v \in \mathrm{Im}(J^{(a)}_{\phi(n)l} Z \to J^{(a)}_{nl} Z)$ be an arbitrary point. We have to show that $v \in \mathrm{Im}(J^{(a)}_{\infty} Z \to J^{(a)}_{nl} Z)$. This holds, either if $v \notin J^{(a)}_{nl} X$ or if $v \in \mathrm{Im}(J^{(a)}_{\psi(n)l} X \to J^{(a)}_{nl} X)$. If it is not the case, there exists $w \in \mathrm{Im}(J^{(a)}_{\phi(n)l} Z \to J^{(a)}_{\psi(n)l} Z) \setminus J^{(a)}_{\psi(n)l} X$ which maps to v. By the definition of τ, we have $w \in \mathrm{Im}(J^{(a)}_{\infty} Z \to J^{(a)}_{\psi(n)l} Z)$ and $v \in \mathrm{Im}(J^{(a)}_{\infty} Z \to J^{(a)}_{nl} Z)$. □

Lemma 3.28 (Equivariant Newton-Hensel lemma)**.** *Let l be a positive integer prime to the characteristic of k. Suppose that k contains all l-th roots of unity. Suppose that $\boldsymbol{\mu}_l$ acts on k-algebras $k[[t]][x_1, \dots, x_d]$ and $k[[t]]$ by*

$$\begin{aligned} \boldsymbol{\mu}_l \ni \zeta : & t \mapsto \zeta t \\ & x_i \mapsto \zeta^{a_i} x_i \ (a_i \in \{0, 1, \dots, l-1\}) \textit{ and} \\ \boldsymbol{\mu}_l \ni \zeta : & t \mapsto \zeta t \textit{ respectively.} \end{aligned}$$

Let Λ be the set of $\boldsymbol{\mu}_l$-equivariant $k[[t]]$-algebra homomorphisms

$$k[[t]][x_1, \dots, x_d] \to k[[t]].$$

Let f_i $(i = 1, 2, \dots, r \le d)$ be elements in the invariant subring $k[[t]][x_1, \dots, x_d]^{\boldsymbol{\mu}_l}$ and let

$$J := \det \left(\frac{\partial f_i}{\partial x_j} \right)_{1 \le i \le r,\ 1 \le j \le r} \in k[[t]][x_1, \dots, x_d].$$

Suppose that for a positive integer e such that $e' := e + \sum_{j=1}^{r} a_j$ is divisible by l and for $\eta \in \Lambda$, we have

$$\eta(J) \not\equiv 0 \mod (t^{e+1})$$

and for $i = 1, 2, \dots, r$,

$$\eta(f_i) \equiv 0 \mod (t^{2e'+l}).$$

Then there is $\theta \in \Lambda$ such that for every i,

$$\theta(f_i) = 0 \textit{ and } \theta(x_i) \equiv \eta(x_i) \mod (t^{e+l}).$$

Proof. Let $\gamma : k[[t]][x_1, \dots, x_d] \to k[[t]]$ be a $k[[t]]$-algebra homomorphism. Then $\gamma \in \Lambda$ if and only if for every i, $\gamma(x_i) \in t^{a_i} \cdot k[[t^l]]$. Consider an $k((t))$-automorphism

$$\alpha : k((t))[x_1, \dots, x_d] \to k((t))[x_1, \dots, x_d],\ x_i \mapsto t^{-a_i} x_i.$$

If $\tilde{\gamma} : k((t))[x_1, \dots, x_d] \to k((t))$ is a $k((t))$-algebra homomorphism which is the extension of $\gamma \in \Lambda$, then for every i, $\tilde{\gamma} \circ \alpha(x_i)$ lies in a

subring $k[[t^l]] \subset k[[t]]$ and hence $\tilde{\gamma} \circ \alpha(k[[t^l]][x_1, \dots, x_d]) \subset k[[t^l]]$. We define a $k[[t^l]]$-algebra homomorphism

$$\gamma' : k[[t^l]][x_1, \dots, x_d] \to k[[t^l]]$$

to be the restriction of $\tilde{\gamma} \circ \alpha$. Let Λ' be the set of $k[[t^l]]$-algebra homomorphisms $k[[t^l]][x_1, \dots, x_d] \to k[[t^l]]$. Then the map

$$\Lambda \to \Lambda', \ \gamma \mapsto \gamma'$$

is bijective.

The invariant ring $k[[t]][x_1, \dots, x_d]^{\boldsymbol{\mu}_l}$ is generated as a $k[[t^l]]$-algebra by the monomials $t^b x_1^{b_1} \dots x_d^{b_d}$ such that $b + \sum_{i=1}^d a_i b_i \equiv 0 \mod l$. Since

$$\alpha^{-1}(t^b x_1^{b_1} \cdots x_d^{b_d}) = t^{b+\sum a_i b_i} x_1^{b_1} \cdots x_d^{b_d} \in k[[t^l]][x_1, \dots, x_d],$$

$f'_i := \alpha^{-1}(f_i)$ lie in $k[[t^l]][x_1, \dots, x_d]$. Let

$$J' := \det \left(\frac{\partial f'_i}{\partial x_j} \right)_{1 \le i \le r, \ 1 \le j \le r}.$$

For any $g \in k((t))[x_1, \dots, x_d]$, by easy calculation, we can see

$$\frac{\partial \alpha^{-1}(g)}{\partial x_i} = t^{a_i} \alpha^{-1} \left(\frac{\partial g}{\partial x_i} \right).$$

Hence we have

$$J' = t^{\sum_{j=1}^r a_j} \alpha^{-1}(J).$$

We can assume that $(\eta'(J')) = (t^{e'})$, or equivalently that $(\eta(J)) = (t^e)$: For if $\eta'(J') = (t^c) \subset k[[t^l]]$, we replace f_1 and f'_1 with $t^{e'-c} f_1$ and $t^{e'-c} f'_1$.

We have

$$\eta'(f'_i) = \eta(f_i) \in (t^{2e'+l}) \subset k[[t^l]].$$

Now from the (non-equivariant) Newton-Hensel lemma [Bou, Chapitre III, §4, Corollaire 3], there exists $\theta' \in \Lambda'$ such that

$$\theta'(f'_i) = 0 \text{ and } \theta'(x_i) \equiv \eta'(x_i) \mod (t^{e'+l}).$$

Let θ be the element mapping to θ' by the bijection above $\Lambda \to \Lambda'$. Then $\theta(f_i) = 0$ for every i. Moreover since $\theta(x_i) = \theta'(x_i) t^{a_i}$ and $\eta(x_i) = \eta'(x_i) t^{a_i}$, we have that for every i,

$$\theta(x_i) \equiv \eta(x_i) \mod (t^{e'+l+a_i}),$$

hence

$$\theta(x_i) \equiv \eta(x_i) \mod (t^{e+l}).$$

□

Lemma 3.29. *Let $\mathcal{Z}$ be a DM stack of finite type and dimension d'.*

(1) *For every* $0 \le n < \infty$, *every fiber of* $\pi_{n+1}(\mathcal{J}_\infty \mathcal{Z}) \to \pi_n(\mathcal{J}_\infty \mathcal{Z})$ *is of dimension* $\le d'$.
(2) $\dim \pi_n(\mathcal{J}_\infty \mathcal{Z}) \le d'(n+1)$.

Proof. The second assertion is a direct consequence of the first. Now we may assume that k is algebraically closed and $\mathcal{Z}$ is a quotient stack $[Z/G]$. From Proposition 2.8, it suffices to show that for every embedding $a : \boldsymbol{\mu}_l \hookrightarrow G$, every closed fiber of a morphism

$$\pi_{(n+1)l}(J_\infty^{(a)} Z) \to \pi_{nl}(J_\infty^{(a)} Z) \tag{3.1}$$

is of dimension $\le d'$. Take a $\boldsymbol{\mu}_l$-equivariant embedding

$$Z \hookrightarrow \mathbb{A}^d = \operatorname{Spec} k[x_1, \dots, x_d]$$

where $\boldsymbol{\mu}_l$ acts on Z through a and on $\mathbb{A}^d$ by

$$\boldsymbol{\mu}_l \ni \zeta : x_i \mapsto \zeta^{a_i} x_i, \ 0 \le a_i \le l-1.$$

Let $\mathbf{f} = (f_1, \dots, f_r)$ be a system of polynomials in $k[x_1, \dots, x_d]$ which defines Z. Then $(J_\infty^{(a)} Z)(k)$ is identified with

$$\{(\phi_1, \dots, \phi_d) | \phi_i \in t^{a_i} \cdot k[[t^l]] \subset k[[t]], \ \mathbf{f}(\phi_1, \dots, \phi_d) = 0\}.$$

Let $\gamma \in J_\infty^{(a)} Z$ correspond to $(\phi_1, \dots, \phi_d)$. Then the fiber of (3.1) over $\pi_{nl}(\gamma)$ is identified with

$$B := \{(\bar{\psi}_1, \dots, \bar{\psi}_d) | \mathbf{f}(\dots, \phi_i + t^{a_i + nl} \psi_i, \dots) = 0, \ \psi_i \in k[[t^l]]\}.$$

Here $\bar{\psi}$ is the image of ψ by $k[[t]] \twoheadrightarrow k = k[[t]]/t$. This is contained in

$$B' := \{(\bar{\psi}_1, \dots, \bar{\psi}_d) | \mathbf{f}(\dots, \phi_i + t^{a_i + nl} \psi_i, \dots) = 0, \ \psi_i \in k[[t]]\}.$$

The equations $\mathbf{f}(\dots, \phi_i + t^{a_i + nl} \psi_i, \dots) = 0$ define a closed subscheme

$$Y \subset \operatorname{Spec} k[[t]][[\psi_1, \dots, \psi_d]].$$

The generic fiber of the projection $Y \to \operatorname{Spec} k[[t]]$ is isomorphic to $Z \otimes_k k((t))$. Then B' is contained in the intersection of the special fiber and the closure of the generic fiber. Hence $\dim B \le \dim B' \le d'$. □

3.6. **Motivic integration over singular varieties.** We review the motivic integration over singular varieties which was studied by Denef and Loeser [DL1]. They assumed that the base field is of characteristic zero. Their arguments however apply to an arbitrary perfect field, as verified by Sebag [Seb] in a more general situation. Although they considered only schemes, we can simply generalize the theory to algebraic spaces.

Comparing a DM stack and its coarse moduli space is an interesting problem. Even if the stack is smooth, the coarse moduli space is not

generally smooth. Therefore, we consider not only the motivic integration over smooth DM stacks, but also that over singular varieties.

Let X be a reduced variety of pure dimension d. If X is not smooth, then fibers of $J_{n+1}X \to J_nX$ are not generally isomorphic to $\mathbb{A}^d$. From Greenberg's theorem [Gre], $\pi_n(J_\infty X)$ is a constructible subset. Denef and Loeser proved that every fiber of $\pi_{n+1}(J_\infty X) \to \pi_n(J_nX)$ is of dimension $\le d$ [DL1, Lemma 4.3].

For an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ and $\gamma \in (J_\infty X)(K)$ with K a field, we define the order of $\mathcal{I}$ along γ to be $\operatorname{ord}\mathcal{I}(\gamma) := n$ if $\gamma^{-1}\mathcal{I} = (t^n) \subset K[[t]]$. By convention, we put $\operatorname{ord}\mathcal{I}(\gamma) := \infty$ if $\gamma^{-1}\mathcal{I} = (0)$. Thus we have the order function associated to $\mathcal{I}$,

$$\operatorname{ord}\mathcal{I} : J_\infty X \to \mathbb{Z}_{\ge 0} \cup \{\infty\}.$$

Let Jac_X be the Jacobian ideal sheaf of X, that is, the d-th Fitting ideal of $\Omega_{X/k}$. We define

$$J_n^{\diamondsuit}X := \{\pi_n(\gamma) | \gamma \in J_\infty X,\ \operatorname{ord}\mathrm{Jac}_X(\gamma) < n\} \subset \pi_n(J_\infty X).$$

This is a constructible subset of J_nX. The fibers of $\pi_{n+1}(J_\infty X) \to \pi_n(J_\infty X)$ over $J_n^{\diamondsuit}X$ are isomorphic to a d-dimensional affine space (see [Loo, Lemma 9.1]).

Definition 3.30. A subset $A \subset J_\infty X$ is called an *n-cylinder* if $A = \pi_n^{-1}\pi_n(A)$ and $\pi_n(A)$ is a constructible subset in $J_n^{\diamondsuit}X$. A subset is called a *cylinder* if it is an n-cylinder for some $n \in \mathbb{Z}_{\ge 0}$. For an n-cylinder A, we define $\operatorname{codim} A := (n+1)d - \dim \pi_n(A)$.

For an n-cylinder A, we define

$$\mu_X(A) := \{\pi_n(A)\}\mathbb{L}^{-nd} \in \mathfrak{R}.$$

As in the case of smooth stacks, we say that a function $F : J_\infty X \supset A \to \mathfrak{R}$ is *measurable* if there are countably many cylinders A_i such that $A = \coprod_i A_i$ and the restriction of F to each A_i is constant. For a measurable function F, we define

$$\int_A F d\mu_{\mathcal{X}} := \sum F(A_i) \cdot \mu_{\mathcal{X}}(A_i) \in \bar{\mathfrak{R}}.$$

Definition 3.31. A subset $A \subset J_\infty X$ is said to be a *negligible* if there are constructible subsets $C_n \subset \pi_n(J_\infty X)$, $n \in \mathbb{N}$ such that $A = \bigcap_{n\in\mathbb{N}} \pi_n^{-1}(C_n)$ and $\lim_{n\to\infty} \dim C_n - dn = -\infty$.

When X is smooth, this definition coincides with that in the preceding subsection. For a subvariety $Z \subset X$ of positive codimension, $J_\infty Z$ is a negligible subset of $J_\infty X$. We indeed have

$$J_\infty Z = J_\infty Z_{\mathrm{red}} = \bigcap_{n\in\mathbb{N}} \pi_n^{-1}\pi_n(J_\infty Z_{\mathrm{red}}),$$

where $Z_{\mathrm{red}} \subset X$ is the reduced subscheme associated to Z. As Denef and Loeser proved, the constructible subset $\pi_n(J_\infty Z_{\mathrm{red}})$ is of dimension $\le (n+1)\dim Z$.

Proposition 3.32. *Let $F : A \to \mathfrak{R}$ be a function and $B \subset A$ a negligible subset. Then if F is measurable, then $F|_{A\setminus B}$ is measurable and*

$$\int_A F d\mu_{\mathcal{X}} = \int_{A\setminus B} F d\mu_{\mathcal{X}}.$$

Proof. Let $E_i := \{\gamma \in |J_\infty X| | \mathrm{ord}\,\mathrm{Jac}_X(\gamma) = i\}$, $i \ge 0$ and let $C_n \subset \pi_n(J_\infty X)$ $n \in \mathbb{N}$ be constructible subsets such that $B = \bigcap_n \pi_n^{-1}(C_n)$ and $\lim_{n\to\infty} \dim C_n - dn = -\infty$. Replacing A and B with $A \cap E_i$ and $B \cap E_i$, we may assume that $A, B \subset E_i$ for some i. Subsets $C'_n := \pi_n^{-1}(C_n) \cap E_i$ are cylinders and $\lim_{n\to\infty} \mathrm{codim}\, C'_n = \infty$. Now we can prove the assertions by the same argument as in the proof of Proposition 3.25. □

Again, by abuse of terminology, we say that F is a measurable function on A even if F is a measurable function defined only on $A \setminus B$ with B negligible.

Let $f : Y \to X$ be a proper birational morphism of reduced varieties of pure dimension and let $X' \subset X$ and $Y' \subset Y$ be proper closed subsets such that $f : Y \setminus Y' \cong X \setminus X'$. Then from the valuative criterion for the properness, the map $f_\infty : J_\infty Y \setminus J_\infty Y' \to J_\infty X \setminus J_\infty X'$ is bijective. In other words, the map $f_\infty : J_\infty Y \to J_\infty X$ is bijective outside negligible subsets. The most fundamental theorem in the theory is the following transformation rule (the change of variables formula) which describes the relation of μ_X and μ_Y. This was by Kontsevich [Kon], Denef and Loeser [DL1], and Sebag [Seb].

Theorem 3.33. *Let $f : Y \to X$ be a proper birational morphism of reduced varieties of pure dimension. Assume that Y is smooth. Let $\mathrm{Jac}_f \subset \mathcal{O}_Y$ be the Jacobian ideal of the morphism f, that is, the 0-th Fitting ideal of $\Omega_{Y/X}$. Let $F : J_\infty X \supset A \to \mathfrak{R}$ be a measurable function. Then F is measurable if and only if $(F \circ f_\infty) \cdot \mathbb{L}^{-\mathrm{ord}\,\mathrm{Jac}_f}$ is measurable. If they are measurable, then we have*

$$\int_A F d\mu_X = \int_{f_\infty^{-1}(A)} (F \circ f_\infty) \cdot \mathbb{L}^{-\mathrm{ord}\,\mathrm{Jac}_f} d\mu_Y.$$

Sketch of the proof. Let $\gamma \in (J_\infty X)(K)$. Suppose that γ sends the generic point of $\mathrm{Spec}\, K[[t]]$ into the locus where f^{-1} is an isomorphism. The theorem is essentially a consequence of the facts that f_∞ is bijective outside negligible subsets and that for $n \gg 0$, the fiber $f_n^{-1}(f_n \circ \pi_n(\gamma))$ is

isomorphic to an affine space of dimension $\operatorname{ord} \operatorname{Jac}_f(\gamma)$. (We generalize this fact to the stack case in Lemma 3.43.) $\square$

3.7. **Tame proper birational morphisms and twisted arcs.** In this subsection, we generalize to the stack case the fact that for a proper birational morphism f of varieties, f_∞ is bijective outside negligible subsets.

Definition 3.34. A morphism $f : \mathcal{Y} \to \mathcal{X}$ of DM stacks is said to be *tame* if for every geometric point y of $\mathcal{Y}$, the kernel of $\operatorname{Aut}(y) \to \operatorname{Aut}(f(y))$ is of order prime to the characteristic of k.

The following are clear.

Lemma 3.35.
(1) *Tame morphisms are stable under base change.*
(2) *Every representable morphism is tame.*
(3) *The composite of tame morphisms is tame.*

Definition 3.36. A morphism $f : \mathcal{Y} \to \mathcal{X}$ of DM stacks is said to be *birational* if there are open dense substacks $\mathcal{Y}_0 \subset \mathcal{Y}$ and $\mathcal{X}_0 \subset \mathcal{X}$ such that f induces an isomorphism $\mathcal{Y}_0 \cong \mathcal{X}_0$.

For example, given an effective action of a finite group G on an irreducible variety M (that is, for $1 \neq g \in G$, $M^g \subsetneqq M$), then the natural morphism from the quotient stack $[M/G]$ to the quotient variety M/G is birational. More generally, the morphism from a DM stack $\mathcal{X}$ to its coarse moduli space is birational if $\mathcal{X}$ contains an open dense substack which is isomorphic to an algebraic space.

Proposition 3.37. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a tame proper birational morphism of DM stacks. Let $\mathcal{Y}' \subset \mathcal{Y}$ and $\mathcal{X}' \subset \mathcal{X}$ be closed substacks such that f induces an isomorphism $\mathcal{Y} \setminus \mathcal{Y}' \cong \mathcal{X} \setminus \mathcal{X}'$. Then the map*

$$f_\infty : |\mathcal{J}_\infty \mathcal{Y}| \setminus |\mathcal{J}_\infty \mathcal{Y}'| \to |\mathcal{J}_\infty \mathcal{X}| \setminus |\mathcal{J}_\infty \mathcal{X}'|$$

is bijective. In particular, if $\mathcal{Y}$ and $\mathcal{X}$ are either a smooth DM stack or a reduced variety of pure dimension, then f_∞ is bijective outside negligible subsets.

Proof. A weak version of this lemma was proved in [Yas1, Lemma 3.17]. A similar argument works in this general case.

Let $K \supset k$ be an algebraically closed field and $\gamma' : \mathcal{D}^{l'}_{\infty,K} \to \mathcal{X}$ a twisted arc such that the generic point maps into $\mathcal{X} \setminus \mathcal{X}'$. Let $\mathcal{E}$ to be the irreducible component of $\mathcal{D}^{l'}_{\infty,K} \times_{\mathcal{X}} \mathcal{Y}$ which contains $\operatorname{Spec} K((t)) \times_{\mathcal{X}} \mathcal{Y}$ and $\mathcal{D}$ the normalization of $\mathcal{E}$. Then the natural morphism $\mathcal{D} \to \mathcal{Y}$ is representable. The stack $\mathcal{D}$ is tame and formally smooth over K, and the natural morphism $\mathcal{D} \to \mathcal{D}^{l'}_{\infty,K}$ is proper and birational.

Hence $\mathcal{D}$ must be isomorphic to $\mathcal{D}^{l}_{\infty,K}$ for some l which is prime to the characteristic of k and a multiple of l'.

Choose the isomorphism $\mathcal{D} \cong \mathcal{D}^{l}_{\infty,K}$ so that the morphism $\mathcal{D}^{l}_{\infty,K} \cong \mathcal{D} \to \mathcal{D}^{l'}_{\infty,K}$ induces the morphism $D_{\infty,K} \to D_{\infty,K}$ of canonical atlases defined by $t \mapsto t^{l/l'}$. Thus we have obtained a twisted arc $\gamma : \mathcal{D}^{l}_{\infty,K} \to \mathcal{Y}$. In fact, this is the unique twisted arc which maps to γ'. It follows from the universalities of the fiber product and the normalization. □

Remark 3.38. This proposition does not hold if we consider only *non-twisted* arcs $\operatorname{Spec} K[[t]] \to \mathcal{X}$. It is why we have to introduce the notion of twisted jets.

3.8. **Fractional Tate objects.** We generalize the transformation rule to proper tame birational morphisms of DM stacks in §3.10. Then the contribution of automorphisms of points appears in the formula. It is of the form $\mathbb{L}^q$ with q a rational number. Therefore we extend the ring in which integrals take values so that it contains fractional powers of $\mathbb{L}$.

We have another motivation to consider fractional powers of $\mathbb{L}$. In the birational geometry, particularly in the minimal model program, we often treat a normal variety X with $\mathbb{Q}$-Cartier canonical divisor (that is, X is $\mathbb{Q}$-Gorenstein) or more generally a pair (X, D) of a normal variety and a $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. We can define invariants of X or (X, D), integrating a function of the form $\mathbb{L}^h$ with h a $\mathbb{Q}$-valued function deriving from K_X or $K_X + D$. We deal with this subject in the context generalized to DM stacks in the final section.

Replacing a $\mathbb{Z}$-valued function α in Definition 3.1 with $\frac{1}{r}\mathbb{Z}$-valued function, we obtain a $\frac{1}{r}\mathbb{Z}$-convergent stack. We define the same equivalence relation $\sim$ of the set $(\mathfrak{R}^{1/r})'$ of the isomorphism classes of $\frac{1}{r}\mathbb{Z}$-convergent stacks. Then we define $\mathfrak{R}^{1/r}$ to be $(\mathfrak{R}^{1/r})'$ modulo $\sim$. This is also a semiring and endowed with a map

$$\dim : \mathfrak{R}^{1/r} \to \frac{1}{r}\mathbb{Z} \cup \{-\infty\}.$$

We denote $\{(\operatorname{Spec} k, 1/r)\}$ by $\mathbb{L}^{1/r}$. Then we have $(\mathbb{L}^{1/r})^r = \mathbb{L}$. We can naturally consider $\mathfrak{R}^{1/r}$-valued measurable functions $F : A \to \mathfrak{R}^{1/r}$ and their integrals:

$$\int_A F d\mu_{\mathcal{X}} = \sum F(A_i)\mu_{\mathcal{X}}(A_i) \in \bar{\mathfrak{R}}^{1/r} := \mathfrak{R}^{1/r} \cup \{\infty\}.$$

We similarly define the $\frac{1}{r}\mathbb{Z}$-convergent space and the semiring $\mathfrak{S}^{1/r}$ of equivalence classes of $\frac{1}{r}\mathbb{Z}$-convergent spaces. There is a semiring homomorphism $\mathfrak{R}^{1/r} \to \mathfrak{S}^{1/r}$, $\{\mathcal{X}\} \mapsto \{\bar{\mathcal{X}}\}$.

We simply define a $\frac{1}{r}\mathbb{Z}$*-indexed Hodge structure* to be a finite dimensional $\mathbb{Q}$-vector space H with a decomposition

$$H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p,q \in \frac{1}{r}\mathbb{Z}} H^{p,q}$$

such that $H^{q,p} = \overline{H^{p,q}}$. Then a $\frac{1}{r}\mathbb{Z}$-indexed mixed Hodge structure is a finite dimensional $\mathbb{Q}$-vector space H endowed with a $\frac{1}{r}\mathbb{Z}$-indexed weight filtration $W_\bullet$ of H and a $\frac{1}{r}\mathbb{Z}$-indexed Hodge filtration $F^\bullet$ of $H \otimes_{\mathbb{Q}} \mathbb{C}$. The associated graded $\bigoplus_{w \in \frac{1}{r}\mathbb{Z}} \mathrm{Gr}_w^W H$ is a $\frac{1}{r}\mathbb{Z}$-indexed Hodge structure. We denote the category of $\frac{1}{r}\mathbb{Z}$-indexed mixed Hodge structures by $MHS^{1/r}$. For $a \in \frac{1}{r}\mathbb{Z}$, the Tate-Hodge structure $\mathbb{Q}(a)$ is defined to be the one-dimensional $\frac{1}{r}\mathbb{Z}$-indexed Hodge structure H such that $H^{-a,-a}$ is the only nonzero component of $H \otimes_{\mathbb{Q}} \mathbb{C}$. We denote $[\mathbb{Q}(a)] \in K_0(MHS^{1/r})$ by $\mathbb{L}^{-a}$. We define also the completion $\hat{K}_0(MHS^{1/r})$ similarly. When $k = \mathbb{C}$, for a $\frac{1}{r}\mathbb{Z}$-convergent space $X = (X, \alpha)$, we define

$$\chi_h(X) := \sum_{V \subset X} \chi_h(V)\mathbb{L}^{\alpha(V)} \in \hat{K}_0(MHS^{1/r}).$$

There are semiring homomorphisms

$$\mathfrak{S}^{1/r} \to \hat{K}_0(MHS^{1/r}),\ \{X\} \mapsto \chi_h(X)$$
$$\mathfrak{R}^{1/r} \to \hat{K}_0(MHS^{1/r}),\ \{\mathcal{X}\} \mapsto \chi_h(\bar{\mathcal{X}}).$$

Suppose that k is a finite field and p is a prime number different from the characteristic of k. If there exist $V \in MR(\mathbf{G}_k, \mathbb{Q}_p)$ such that $V^{\otimes r} \cong \mathbb{Q}_p(1)$, then we fix V and denote it by $\mathbb{Q}_p(1/r)$. This is pure of weight $-2/r$. For a positive integer a, we define $\mathbb{Q}_p(a/r) := \mathbb{Q}_p(1/r)^{\otimes a}$ and $\mathbb{Q}_p(-a/r)$ to be its dual. We denote an element $[\mathbb{Q}_p(-1/r)] \in K_0(MR(\mathbf{G}_k, \mathbb{Q}_p))$ also by $\mathbb{L}^{1/r}$.

T. Ito [Ito2] proved that if we replace k with its suitable finite extension, then $\mathbb{Q}_p(1/r)$ exists. He used this to give a new proof of the well-definedness of stringy Hodge numbers with p-adic integrals and the p-adic Hodge theory.

When $\mathbb{L}^{1/r}$ exists, for a $\frac{1}{r}\mathbb{Z}$-convergent space $X = (X, \alpha)$, we define

$$\chi_p(X) := \sum_{V \subset X} \chi_p(V)\mathbb{L}^{\alpha(V)} \in \hat{K}_0(MR(\mathbf{G}_k, \mathbb{Q}_p)).$$

Then we have semiring homomorphisms

$$\mathfrak{S}^{1/r} \to \hat{K}_0(MR(\mathbf{G}_k, \mathbb{Q}_p)),\ \{X\} \mapsto \chi_p(X)$$
$$\mathfrak{R}^{1/r} \to \hat{K}_0(MR(\mathbf{G}_k, \mathbb{Q}_p)),\ \{\mathcal{X}\} \mapsto \chi_h(\bar{\mathcal{X}}).$$

3.9. **Shift number.** Let $\mathcal{X}$ be a smooth DM stack of pure dimension d and $x \in \mathcal{X}(K)$ a geometric point. Then the automorphism group $\mathrm{Aut}(x)$ linearly acts on the tangent space $T_x\mathcal{X}$. Let $a : \boldsymbol{\mu}_l \hookrightarrow \mathrm{Aut}(x)$ be an embedding. According to the $\boldsymbol{\mu}_l$-action, $T_x\mathcal{X}$ decomposes into eigenspaces;

$$T_x\mathcal{X} = \bigoplus_{i=1}^{l} T_{i,x}.$$

Here $T_{i,x}$ is the eigenspace on which $\zeta \in \boldsymbol{\mu}_l$ acts by the multiplication of ζ^i. We define

$$\mathrm{sht}(a) := d - \frac{1}{l}\sum_{i=1}^{l} i \cdot \dim T_{i,x} = \frac{1}{l}\sum_{i=1}^{l}(l-i)\cdot \dim T_{i,x}.$$

If a' is a conjugacy of a, then $\mathrm{sht}(a') = \mathrm{sht}(a)$. Since the fiber of $|\mathcal{J}_0\mathcal{X}| \to |\mathcal{X}|$ over x is identified with $\coprod_{\mathrm{char}\,(k)\nmid l} \mathrm{Conj}(\boldsymbol{\mu}_l, \mathrm{Aut}(x))$, we define $\mathrm{sht}(p) := \mathrm{sht}(a)$ if $p \in |\mathcal{J}_0\mathcal{X}|$ is the point corresponding to (x, a). Thus we have a map

$$\mathrm{sht} : |\mathcal{J}_0\mathcal{X}| \to \mathbb{Q}.$$

Furthermore, $\mathrm{sht}(p)$ depends only on the connected component $\mathcal{V} \subset \mathcal{J}_0\mathcal{X}$ in which p lies: Let $f : S \to \mathcal{V}$ be a morphism with S a connected scheme such that $p : \mathrm{Spec}\,K \to \mathcal{V}$ factors as $\mathrm{Spec}\,K \to S \to \mathcal{V}$. Let $f' : S \to \mathcal{V} \to \mathcal{X}$ be the composite of f and the projection $\mathcal{V} \to \mathcal{X}$. Then the pull-back $(f')^*T\mathcal{X}$ of the tangent bundle has a $\boldsymbol{\mu}_{l,k}$-action naturally deriving from f and decomposes into eigenbudles,

$$(f')^*T\mathcal{X} = \bigoplus_{i=1}^{l} T_i$$

such that $T_{i,x}$ above is a pull-back of T_i. Then

$$\mathrm{sht}(p) = d - \frac{1}{l}\sum_{i=1}^{l} i \cdot \mathrm{rank}\, T_i.$$

It follows that $\mathrm{sht}(p)$ is constant on $\mathcal{V}$. We define $\mathrm{sht}(\mathcal{V}) := \mathrm{sht}(p)$, $p \in |\mathcal{V}|$.

We denote the composite map $|\mathcal{J}_\infty\mathcal{X}| \to |\mathcal{J}_0\mathcal{X}| \xrightarrow{\mathrm{sht}} \mathbb{Q}$ by $\mathfrak{s}_\mathcal{X}$. For a (possibly singular) variety X, we denote by $\mathfrak{s}_X$ the constant zero function over $|J_\infty X|$. These definitions coincide for smooth varieties. In both cases, the function

$$\mathbb{L}^{\mathfrak{s}_\mathcal{X}} : |\mathcal{J}_\infty\mathcal{X}| \to \mathfrak{R}^{1/r}$$

is clearly measurable.

3.10. **Transformation rule.** In this subsection, we prove the transformation rule generalized to tame proper birational morphisms.

Definition 3.39. Let $\mathcal{X}$ be a DM stack and $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ an ideal sheaf. Let $\mathcal{Z} \subset \mathcal{X}$ be the closed substack defined by $\mathcal{I}$. We define a function

$$\operatorname{ord} \mathcal{I} : |\mathcal{J}_\infty \mathcal{X}| \setminus |\mathcal{J}_\infty \mathcal{Z}| \to \mathbb{Q}$$

as follows: Let $\gamma \in |\mathcal{J}_\infty \mathcal{X}|$ and $\gamma_K : \mathcal{D}^l_{\infty,K} \to \mathcal{X}$ its representative. Let $\gamma'_K : D_{\infty,K} \to \mathcal{X}$ be the composite of γ_K and the canonical atlas $D_{\infty,K} \to \mathcal{D}^l_{\infty,K}$. Suppose that $(\gamma'_K)^{-1}\mathcal{I} = (t^m) \subset K[[t]]$. Then

$$\operatorname{ord} \mathcal{I}(\gamma) := \frac{m}{l} \in \mathbb{Q}.$$

Let $\mathcal{X}$ be a smooth DM stack and $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ an ideal sheaf such that the support of $\mathcal{O}_{\mathcal{X}}/\mathcal{I}$ is of positive codimension. Let $r \in \mathbb{N}$ be such that $\operatorname{Im}(\operatorname{ord} \mathcal{I}) \subset \frac{1}{r}\mathbb{Z}$. Then the function

$$\mathbb{L}^{\operatorname{ord} \mathcal{I}} : |\mathcal{J}_\infty \mathcal{X}| \to \mathfrak{R}^{1/r}$$

is measurable (defined outside a negligible subset $|\mathcal{J}_\infty \mathcal{Z}|$).

Definition 3.40. Let $f : \mathcal{Y} \to \mathcal{X}$ be a birational morphism of DM stacks. We define its *Jacobian ideal sheaf* $\operatorname{Jac}_f \subset \mathcal{O}_{\mathcal{X}}$ to be the 0-th Fitting ideal sheaf of $\Omega_{\mathcal{Y}/\mathcal{X}}$.

Theorem 3.41 (Transformation rule)**.** *Let $\mathcal{Y}$ and $\mathcal{X}$ be DM stacks of finite type and pure dimension d and $f : \mathcal{Y} \to \mathcal{X}$ a tame proper birational morphism. Suppose that $\mathcal{Y}$ is smooth and $\mathcal{X}$ is either a smooth DM stack or a reduced variety. Let $A \subset |\mathcal{J}_\infty \mathcal{X}|$ be a subset and $F : A \to \mathfrak{R}^{1/r}$ a function (at least, defined outside a negligible subset).*

(1) *The function F is measurable if and only if $F \circ f_\infty$ is measurable.*
(2) *Suppose that F is measurable and that the function $\mathbb{L}^{\mathfrak{s}_{\mathcal{X}}}|_A$ takes values in $\mathfrak{R}^{1/r}$. Then*

$$\int_A F\mathbb{L}^{\mathfrak{s}_{\mathcal{X}}} d\mu_{\mathcal{X}} = \int_{f_\infty^{-1}(A)} (F \circ f_\infty)\mathbb{L}^{-\operatorname{ord} \operatorname{Jac}_f + \mathfrak{s}_{\mathcal{Y}}} d\mu_{\mathcal{Y}} \ \in \bar{\mathfrak{R}}^{1/r}.$$

Proof. 1. Let $\mathcal{X}' \subset \mathcal{X}$ be the closed substack over which f is not an isomorphism. Let $A_i \subset |\mathcal{J}_\infty \mathcal{X}| \setminus |\mathcal{J}_\infty \mathcal{X}'|$, $i \in \mathbb{N}$ be subsets such that $\coprod_{i \in \mathbb{N}} A_i = A \setminus |\mathcal{J}_\infty \mathcal{X}'|$ and F is constant over each A_i. Let $B_i := f_\infty^{-1}(A_i)$. From Lemma 3.44, A_i is a cylinder if and only if B_i is a cylinder. Thus F is measurable if and only if $F \circ f_\infty$ is measurable.

2. Let A_i and B_i be as above. If necessary, taking a refinement of $\{A_i\}_i$, we may assume that $\mathfrak{s}_{\mathcal{X}}$ is constant on every A_i and that $\mathfrak{s}_{\mathcal{Y}}$ and

$\operatorname{ord} \operatorname{Jac}_f$ are constant on every B_i. Then from Lemma 3.44, we have

$$\int_{A_i} F\mathbb{L}^{\mathfrak{s}_{\mathcal{X}}} d\mu_{\mathcal{X}} = \int_{B_i} (F \circ f_\infty)\mathbb{L}^{-\operatorname{ord} \operatorname{Jac}_f + \mathfrak{s}_{\mathcal{Y}}} d\mu_{\mathcal{Y}}.$$

Hence

$$\begin{aligned}
&\int_{A} F\mathbb{L}^{\mathfrak{s}_{\mathcal{X}}} d\mu_{\mathcal{X}} \\
&= \sum_{i \in \mathbb{N}} \int_{A_i} F\mathbb{L}^{\mathfrak{s}_{\mathcal{X}}} d\mu_{\mathcal{X}} \\
&= \sum_{i \in \mathbb{N}} \int_{B_i} (F \circ f_\infty)\mathbb{L}^{-\operatorname{ord} \operatorname{Jac}_f + \mathfrak{s}_{\mathcal{Y}}} d\mu_{\mathcal{Y}} \\
&= \int_{f_\infty^{-1}(A)} (F \circ f_\infty)\mathbb{L}^{-\operatorname{ord} \operatorname{Jac}_f + \mathfrak{s}_{\mathcal{Y}}} d\mu_{\mathcal{Y}}.
\end{aligned}$$

□

3.10.1. *Key lemmas.* Let $\mathcal{X}$ be a DM stack of pure dimension d. We define the Jacobian ideal sheaf $\operatorname{Jac}_{\mathcal{X}}$ to be the d-th Fitting ideal of $\Omega_{\mathcal{X}/k}$. If $\mathcal{X}$ is smooth, then $\operatorname{Jac}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}$. If $\mathcal{X}$ is reduced, then since k is perfect, $\mathcal{X}$ is generically smooth. It follows that $\Omega_{\mathcal{X}/k}$ is generically free of rank d and the support of $\mathcal{O}_{\mathcal{X}}/\operatorname{Jac}_{\mathcal{X}}$ is of positive codimension.

Let $x \in \mathcal{X}(K)$. Since for every $0 \le n \le \infty$, the natural morphism $\mathcal{J}_n\mathcal{X} \to \mathcal{X}$ is representable and affine, the fiber $(\mathcal{J}_n\mathcal{X})_x := \mathcal{J}_n\mathcal{X} \times_{\mathcal{X},x} \operatorname{Spec} K$ is an affine scheme.

The following lemmas are generalizations of [DL2, Lemmas 1.17 and 3.5]. In proving these, we adopt ring-theoretic arguments as in [Loo].

Lemma 3.42. *Let $f : \mathcal{Y} \to \mathcal{X}$ be as in Theorem 3.41. Let $\beta_0, \beta_1 : D_{\infty,K} \to \mathcal{Y}$ be (non-twisted) arcs over a K-point $y \in \mathcal{Y}(K)$, that is, $\beta_0, \beta_1 \in (J_\infty\mathcal{Y})_y(K) = (\mathcal{J}_\infty^1\mathcal{Y})_y(K)$. Assume that $\pi_n f_\infty(\beta_0) = \pi_n f_\infty(\beta_1)$ for some $n \in \mathbb{Z}_{\ge 0}$, and*

$$\begin{aligned}
e &:= \operatorname{ord} \operatorname{Jac}_f(\beta_1) < n, \\
b &:= \operatorname{ord} \operatorname{Jac}_{\mathcal{X}}(f_\infty(\beta_1)) < n.
\end{aligned}$$

Then $\pi_{n-e}(\beta_0) = \pi_{n-e}(\beta_1) \in (J_{n-e}\mathcal{Y})_y(K)$.

Proof. Let $x := f(y)$. From the assumption, the homomorphisms

$$(f\beta_0)^*, (f\beta_1)^* : \mathcal{O}_{\mathcal{X},x} \to K[[t]]$$

are identical modulo $\mathfrak{m}^{n+1}$. Here $\mathcal{O}_{\mathcal{X},x}$ denotes the complete local ring at x and $\mathfrak{m} := (t)$ is the maximal ideal. Hence the homomorphisms

$(f\beta_0)^*$ and $(f\beta_1)^*$ give the same $\mathcal{O}_{\mathcal{X},x}$-module structure to the $K[[t]]$-module $\mathfrak{m}^{n+1}/\mathfrak{m}^{2(n+1)}$. The map

$$\mathcal{O}_{\mathcal{X},x} \xrightarrow{(f\beta_1)^* - (f\beta_0)^*} \mathfrak{m}^{n+1} \twoheadrightarrow \mathfrak{m}^{n+1}/\mathfrak{m}^{2(n+1)}$$

is a k-derivation, which induces a $K[[t]]$-module homomorphism

$$\delta(f\beta_1, f\beta_0) : (f\beta_1)^*\Omega_{\mathcal{X}/k} \to \mathfrak{m}^{n+1}/\mathfrak{m}^{2(n+1)} \twoheadrightarrow \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}.$$

This annihilates the torsion part of $(f\beta_1)^*\Omega_{\mathcal{X}/k}$: Since the d-th Fitting ideal of $(f\beta_1)^*\Omega_{\mathcal{X}/k}$ is (t^b), the torsion part of $(f\beta_1)^*\Omega_{\mathcal{X}/k}$ is of length b. Hence the image of the torsion part in $\mathfrak{m}^{n+1}/\mathfrak{m}^{2(n+1)}$ is contained in $\mathfrak{m}^{2(n+1)-b}/\mathfrak{m}^{2(n+1)}$. Since $n > b$, we have

$$2(n+1) - b > n + 2.$$

Thus $\mathfrak{m}^{2(n+1)-b}/\mathfrak{m}^{2(n+1)}$ is killed by $\mathfrak{m}^{n+1}/\mathfrak{m}^{2(n+1)} \twoheadrightarrow \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$.

Consider an exact sequence of $K[[t]]$-modules

$$(f\beta_1)^*\Omega_{\mathcal{X}/k} \to (\beta_1)^*\Omega_{\mathcal{Y}/k} \to (\beta_1)^*\Omega_{\mathcal{Y}/\mathcal{X}} \to 0.$$

Since $(\beta_1)^*\Omega_{\mathcal{Y}/\mathcal{X}}$ is of length e, $\delta(f\beta_1, f\beta_0)$ lifts to a homomorphism

$$\delta(\beta_1, \beta_2) : (\beta_1)^*\Omega_{\mathcal{Y}/k} \to \mathfrak{m}^{n-e+1}/\mathfrak{m}^{n+2},$$

which derives from

$$(\beta_1)^* - (\beta_2)^* : \mathcal{O}_{\mathcal{Y},y} \to \mathfrak{m}^{n-e+1}/\mathfrak{m}^{n+2}$$

for some $\beta_2 \in (J_\infty\mathcal{Y})_y(K)$. Then

$$\beta_1 \equiv \beta_2 \mod \mathfrak{m}^{n-e+1}, \text{ and}$$
$$f\beta_0 \equiv f\beta_2 \mod \mathfrak{m}^{n+2}.$$

Applying the argument above to β_0 and β_2, we obtain β_3 such that

$$\beta_2 \equiv \beta_3 \mod \mathfrak{m}^{n-e+2}$$
$$(\text{hence } \beta_1 \equiv \beta_3 \mod \mathfrak{m}^{n-e+1}), \text{ and}$$
$$f\beta_0 \equiv f\beta_3 \mod \mathfrak{m}^{n+3}.$$

Repeating this, we obtain a sequence β_i, $i \in \mathbb{N}$ such that

$$\beta_1 \equiv \beta_i \mod \mathfrak{m}^{n-e+1}, \text{ and}$$
$$f\beta_0 \equiv f\beta_i \mod \mathfrak{m}^{n+i}.$$

If we put β_∞ to be the limit of this sequence, we have

$$\beta_1 \equiv \beta_\infty \mod \mathfrak{m}^{n-e+1} \text{ and}$$
$$f\beta_0 = f\beta_\infty.$$

Since f is birational and separated, from the valuative criterion [LMB, Proposition 7.8], β_0 and β_∞ are actually the same. It follows that $\pi_{n-e}(\beta_0) = \pi_{n-e}(\beta_1)$. □

Lemma 3.43. *Let $f : \mathcal{Y} \to \mathcal{X}$ be as in Theorem 3.41. Let $K \supset k$ be an algebraically closed field, $y \in \mathcal{Y}(K)$ and $\beta \in (\mathcal{J}^l_\infty \mathcal{Y})_y(K)$. Suppose that* $\operatorname{ord} \operatorname{Jac}_f(\gamma) = e < \infty$*, $s_1 := \mathfrak{s}_{\mathcal{Y}}(\beta)$, $s_2 := \mathfrak{s}_{\mathcal{X}}(f_\infty \beta)$. Assume that $m := n - \lceil e \rceil > e$ and* $\operatorname{ord} \operatorname{Jac}_{\mathcal{X}}(\beta) < n$*. Then $f_n^{-1} f_n \pi_n(\beta) \cong \mathbb{A}_K^{e-s_1+s_2}$.*

Proof. Let $\beta' : D_{\infty,K} \to \mathcal{Y}$ be the composite of β and the canonical atlas $D_{\infty,K} \to \mathcal{D}^l_{\infty,K}$. Let $m := n - \lceil e \rceil$ and let $\beta_1 \in (\mathcal{J}^l_\infty \mathcal{Y})_y(K)$ be a twisted arc with $\pi_m(\beta) = \pi_m(\beta_1)$. Then as in Lemma 3.42, we obtain a $\boldsymbol{\mu}_l$-equivariant derivation

$$(\beta')^* - (\beta'_1)^* : \mathcal{O}_{\mathcal{Y},y} \to \mathfrak{m}^{ml+1}/\mathfrak{m}^{nl+1}$$

and an associated $\boldsymbol{\mu}_l$-equivariant homomorphism

$$\delta(\beta_1) : (\beta')^* \Omega_{\mathcal{Y}/k} \to \mathfrak{m}^{ml+1}/\mathfrak{m}^{nl+1}$$

Let $\beta_2 \in \mathcal{J}^l_\infty \mathcal{Y}$ be another twisted arc with $\pi_m(\beta) = \pi_m(\beta_2)$. Then $\delta(\beta_1) = \delta(\beta_2)$ if and only if $\pi_n(\beta_1) = \pi_n(\beta_2)$. Therefore if π^n_m denotes the natural morphism $\mathcal{J}_n \mathcal{Y} \to \mathcal{J}_{n'} \mathcal{Y}$, then we can regard $(\pi^n_m)^{-1}(\pi_m(\beta))$ as a subset of $\operatorname{Hom}^{\boldsymbol{\mu}_l}_{K[[t]]}((\beta')^* \Omega_{\mathcal{Y}/k}, \mathfrak{m}^{ml+1}/\mathfrak{m}^{nl+1})$. Here $\operatorname{Hom}^{\boldsymbol{\mu}_l}_{K[[t]]}(,)$ is the set of $\boldsymbol{\mu}_l$-equivariant $K[[t]]$-homomorphisms. In fact, the dimensions of $(\pi^n_m)^{-1}(\pi_m(\beta))$ and $\operatorname{Hom}^{\boldsymbol{\mu}_l}_{K[[t]]}((\beta')^* \Omega_{\mathcal{Y}/k}, \mathfrak{m}^{ml+1}/\mathfrak{m}^{nl+1})$ are equal, and we can identify the two spaces.

Let $x := f(y)$ and $F \subset \operatorname{Aut}(y)$ the largest subgroup acting on $\mathcal{O}_{\mathcal{Y},y}$ trivially. Then since f is birational, the natural map $F \hookrightarrow \operatorname{Aut}(y) \to \operatorname{Aut}(x)$ is injective. Let $b : \boldsymbol{\mu}_l \hookrightarrow \operatorname{Aut}(y)$ and $a : \boldsymbol{\mu}_{l'} \hookrightarrow \operatorname{Aut}(x)$ be embeddings deriving from β and $f_\infty(\beta)$ respectively. We have the following commutative diagram:

$$\begin{tikzcd} \zeta \arrow[d, mapsto] & \in \boldsymbol{\mu}_l \arrow[r, hook, "b"] \arrow[d, two heads] & \operatorname{Aut}(y) \arrow[d] \\ \zeta^{l/l'} & \in \boldsymbol{\mu}_{l'} \arrow[r, hook, "a"'] & \operatorname{Aut}(x). \end{tikzcd}$$

Let $\mathcal{Y}' \subset \mathcal{Y}$ be the closed substack where f is not an isomorphism. From the assumption that $m > e$, $(\pi^n_m)^{-1}\pi_m(\beta) \cap (\mathcal{J}_n \mathcal{Y}')_y = \emptyset$. Hence the automorphism groups of any geometric point of $(\pi^n_m)^{-1}\pi_m(\beta)$ is the centralizer of b in F. It follows that the morphism $\mathcal{J}_n \mathcal{Y} \to \mathcal{J}_n \mathcal{X}$ is representable around $(\pi^n_m)^{-1}\pi_m(\beta)$. We can regard $f_n^{-1} f_n \pi_n(\beta)$ as a subspace of $(\pi^n_m)^{-1}\pi_m(\beta) \cong \mathbb{A}_K^{(n-m)d}$.

Consider the natural homomorphism

$$\mathrm{Hom}^{\boldsymbol{\mu}_l}_{K[[t]]}((\beta')^*\Omega_{\mathcal{Y}/k}, \mathfrak{m}^{ml+1}/\mathfrak{m}^{nl+1}) \to \mathrm{Hom}_{K[[t]]}((f\beta')^*\Omega_{\mathcal{X}/k}, \mathfrak{m}^{ml+1}/\mathfrak{m}^{nl+1}). \tag{3.2}$$

Homomorphisms $\delta(\beta_1)$ and $\delta(\beta_2)$ maps to the same element by this homomorphism if and only if $f_n\pi_n(\beta_1) = f_n\pi_n(\beta_2)$. Hence $f_n^{-1}f_n\pi_n(\beta)$ is isomorphic to the kernel of (3.2).

Choose an isomorphism $\mathcal{O}_{\mathcal{Y},y} \cong K[[y_1,\dots,y_d]]$ such that by the induced $\boldsymbol{\mu}_l$-action on $K[[y_1,\dots,y_d]]$, $\zeta \in \boldsymbol{\mu}_l$ sends y_i to $\zeta^{b_i}y_i$, $1 \le b_i \le l$. Then

$$s_1 = \frac{1}{l}\sum_{i=1}^{d}(l - b_i).$$

We have an isomorphism $(\beta')^*\Omega_{\mathcal{Y}/k} \cong \bigoplus_i K[[t]]dy_i$. We define a $K[[t^l]]$-homomorphism

$$\psi : \bigoplus_i K[[t^l]]dy_i \to (\beta')^*\Omega_{\mathcal{Y}/k},$$

by $\psi(dy_i) := t^{l-b_i}dy_i$. The image of this monomorphism is the submodule of the $\boldsymbol{\mu}_l$-invariant elements. Let $\mathfrak{n} := (t^l)$ be the maximal ideal of $K[[t^l]]$. Then the map deriving from ψ,

$$\mathrm{Hom}^{\boldsymbol{\mu}_l}_{K[[t]]}((\beta')^*\Omega_{\mathcal{Y}/k}, \mathfrak{m}^{ml+1}/\mathfrak{m}^{nl+1}) \to \mathrm{Hom}_{K[[t^l]]}(\bigoplus_i K[[t^l]]dy_i, \mathfrak{n}^{m+1}/\mathfrak{n}^{n+1}),$$

is bijective.

If $\mathcal{X}$ is smooth, we choose an isomorphism $\mathcal{O}_{\mathcal{X},x} \cong K[[x_1,\dots,x_d]]$ such that by the induced $\boldsymbol{\mu}_{l'}$-action on $K[[x_1,\dots,x_d]]$, $\zeta \in \boldsymbol{\mu}_{l'}$ sends x_i to $\zeta^{a_i}x_i$, $1 \le a_i \le l'$. Then we have

$$s_2 = \frac{1}{l'}\sum_i (l' - a_i) = \frac{1}{l}\sum (l - a_i l/l').$$

We have an isomorphism $(f\beta')^*\Omega_{\mathcal{X}/k} = \bigoplus_i K[[t]]dx_i$. We define a $K[[t^l]]$-homomorphism

$$\phi : \bigoplus_i K[[t^l]]dx_i \to (f\beta')^*\Omega_{\mathcal{X}/k},$$

by $\phi(dx_i) := t^{l-a_il/l'}dx_i$. Next, suppose that $\mathcal{X}$ is a variety. Then $(f\beta')^*\Omega_{\mathcal{X}/k} \cong (\bigoplus_{i=1}^d K[[t]]dx_i) \oplus (tors)$ with dx_i symbols, where $(tors)$ denotes the torsion part. We define a $K[[t^l]]$-homomorphism

$$\phi : \bigoplus_i K[[t^l]]dx_i \to (\bigoplus_{i=1}^{d} K[[t]]dx_i) \oplus (tors) \cong (f\beta')^*\Omega_{\mathcal{X}/k},$$

by $dx_i \mapsto dx_i$. In both cases, the image of ϕ is contained in the submodule of the $\boldsymbol{\mu}_l$-invariant elements. Here $\boldsymbol{\mu}_l$ acts on $(f\beta')^*\Omega_{\mathcal{X}/k}$ through $\boldsymbol{\mu}_l \twoheadrightarrow \boldsymbol{\mu}_{l'}$. Since the natural homomorphism $(f\beta')^*\Omega_{\mathcal{X}/k} \to (\beta')^*\Omega_{\mathcal{Y}/k}$ is $\boldsymbol{\mu}_l$-equivariant, the image of the composite map

$$\bigoplus_i K[[t^l]]dx_i \xrightarrow{\phi} (f\beta')^*\Omega_{\mathcal{X}/k} \to (\beta')^*\Omega_{\mathcal{Y}/k}$$

is contained in the submodule of $\boldsymbol{\mu}_l$-invariant elements and the map lifts to

$$\tau : \bigoplus_i K[[t^l]]dx_i \to \bigoplus_i K[[t^l]]dy_i.$$

Consider the homomorphism induced by τ,

$$\mathrm{Hom}(\bigoplus_i k[[t^l]]dy_i, \mathfrak{n}^{m+1}/\mathfrak{n}^{n+1}) \to \mathrm{Hom}(\bigoplus_i k[[t^l]]dx_i, \mathfrak{n}^{m+1}/\mathfrak{n}^{n+1}).$$

Its kernel is isomorphic to the kernel of (3.2) and to

$$\mathrm{Hom}(\mathrm{Coker}\,\tau, \mathfrak{n}^{m+1}/\mathfrak{n}^{n+1}).$$

Hence we have

$$\begin{aligned}
&\dim f_n^{-1} f_n \pi_n(\beta) \\
&= \dim \mathrm{Hom}(\mathrm{Coker}\,\tau, \mathfrak{n}^{m+1}/\mathfrak{n}^{n+1}) \\
&= \dim \mathrm{Coker}\,\tau \\
&= \frac{1}{l}((\dim(\beta')^*\Omega_{\mathcal{Y}/\mathcal{X}}) - \sum_i (l - b_i) + \sum_i (l - a_i l/l')) \\
&= e - s_1 + s_2.
\end{aligned}$$

Thus we have proved Lemma 3.43. ☐

Lemma 3.44. *Let $B \subset \mathcal{J}_\infty \mathcal{Y}$ be a subset. Suppose that* $\mathrm{ord}\,\mathrm{Jac}_f|_B \equiv e < \infty$, $\mathfrak{s}_{\mathcal{Y}}|_B \equiv s_1$ *and* $\mathfrak{s}_{\mathcal{X}}|_{f_\infty(B)} \equiv s_2$. *Then B is a cylinder if and only if $f_\infty(B)$ is so. If B is a cylinder, then we have*

$$\mu_{\mathcal{Y}}(B) = \mu_{\mathcal{X}}(f_\infty(B))\mathbb{L}^{e-s_1+s_2}.$$

Proof. From Proposition 3.37, the assumption that $\mathrm{ord}\,\mathrm{Jac}_f|_B < \infty$ means that the map $B \to f_\infty(B)$ is bijective. Since we have assumed that $\mathcal{Y}$ is smooth, f is not an isomorphism all over the singular locus $\mathcal{X}_{\mathrm{sing}}$ of $\mathcal{X}$. Hence $f_\infty(B)$ lies outside $|\mathcal{J}_\infty \mathcal{X}_{\mathrm{sing}}|$. In particular, $\mathrm{ord}\,\mathrm{Jac}_{\mathcal{X}}$ takes finite values on $f_\infty(B)$. Either if B is a cylinder or if $f_\infty(B)$ is cylinder, then $\mathrm{ord}\,\mathrm{Jac}_{\mathcal{X}}|_{f_\infty(B)}$ is bounded from above: In the case where B is a cylinder, consider $(\mathrm{ord}\,\mathrm{Jac}_{\mathcal{X}}) \circ f_\infty = \mathrm{ord}\, f^{-1}(\mathrm{Jac}_{\mathcal{X}})$.

If $f_\infty(B)$ is an n-cylinder for $n \gg 0$, then for any point $q \in \pi_n(B)$,

$$\pi_n^{-1}(q) \subset \pi_n^{-1} f_n^{-1} f_n(q) \subset f_\infty^{-1} \pi_n^{-1} f_n(q) \subset f_\infty^{-1} f_\infty(B) = B.$$

Since $f_n^{-1}B$ is a constructible subset, it is an n-cylinder.

Next, assume that B is an $(n - \lceil e \rceil)$-cylinder and that n is large enough to satisfy the condition in Lemma 3.42. From Lemma 3.42, for a point $p \in f_n \pi_n(B)$, $\pi_{n-\lceil e \rceil} f_\infty^{-1} \pi_n^{-1}(p)$ is one point. Therefore we have

$$f_\infty^{-1} \pi_n^{-1}(p) \subset B$$

and

$$\pi_n^{-1}(p) \subset f_\infty(B).$$

From Chevalley's theorem [LMB, Théorème 5.9.4], $f_n \pi_n(B)$ is a constructible subset and hence $f_\infty(B)$ is an n-cylinder.

If B is cylinder, from Lemma 3.43, we have

$$\mu_{\mathcal{Y}}(B) = \mu_{\mathcal{X}}(f_\infty(B)) \mathbb{L}^{e-s_1+s_2}.$$

□

4. Birational geometry of Deligne-Mumford stacks

In this section, DM stacks are supposed to be reduced and of finite type.

4.1. **Divisors and invariants of pairs.** Let $\mathcal{X}$ be a DM stack. A *prime divisor* on $\mathcal{X}$ is just a reduced closed substack of $\mathcal{X}$ of codimension one. A *divisor* (resp. $\mathbb{Q}$-*divisor*) is a linear combination of prime divisors with integer (resp. rational) coefficients. A divisor is said to be *Cartier* if the corresponding divisor on an atlas of $\mathcal{X}$ is a Cartier divisor. A divisor D or a $\mathbb{Q}$-divisor is said to be $\mathbb{Q}$-*Cartier* if there exists a positive integer m such that mD is Cartier. If $\mathcal{X}$ is smooth, a divisor (resp. $\mathbb{Q}$-divisor) is always Cartier (resp. $\mathbb{Q}$-Cartier). For a Cartier divisor D, we can define an invertible sheaf $\mathcal{O}_{\mathcal{X}}(D)$ as follows: For an étale morphism $U \to \mathcal{X}$ with U scheme, we put $\mathcal{O}_{\mathcal{X}}(D)_U := \mathcal{O}_U(D_U)$. Here D_U is the pull-back of D to U. The pull-back of a Cartier or $\mathbb{Q}$-Cartier divisor by a morphism is defined in a obvious way.

Now suppose that $\mathcal{X}$ is smooth of pure dimension d. We associate to each $\mathbb{Q}$-divisor D on $\mathcal{X}$ a measurable function $\mathfrak{I}_D : |\mathcal{J}_\infty \mathcal{X}| \to \mathbb{Q}$ as follows: If D is a prime divisor and if $\mathcal{I}_D$ is the defining ideal sheaf of D, then we define

$$\mathfrak{I}_D := \operatorname{ord} \mathcal{I}_D.$$

For a general D, if we write $D = \sum u_i D_i$ with D_i prime divisor and $u_i \in \mathbb{Q} \setminus \{0\}$, then we define

$$\mathfrak{I}_D := \sum_i u_i \mathfrak{I}_{D_i}.$$

This function is defined outside a negligible subset $|\mathcal{J}_\infty(\bigcup D_i)|$. For $\mathbb{Q}$-divisors D and E, we have $\mathfrak{I}_{D+E} = \mathfrak{I}_D + \mathfrak{I}_E$ at least outside a negligible subset.

Definition 4.1. Let $\mathcal{X}$ be a smooth DM stack, D a $\mathbb{Q}$-divisor of $\mathcal{X}$ and $W \subset |\mathcal{X}|$ a constructible subset. Then we define an invariant

$$\Sigma_W(\mathcal{X}, D) := \int_{\pi^{-1}(W)} \mathbb{L}^{\mathfrak{I}_D + \mathfrak{s}_\mathcal{X}} d\mu_\mathcal{X} \in \bar{\mathfrak{R}}^{1/r}.$$

Here $\pi : |\mathcal{J}_\infty \mathcal{X}| \to |\mathcal{X}|$ is the natural projection.

The invariant lies in $\bar{\mathfrak{R}}^{1/r}$ for a *suitable* r. Below, we will take a suitable r each time and not mention it hereinafter.

For a smooth DM stack $\mathcal{X}$, the *canonical sheaf* $\omega_\mathcal{X}$ is defined to be the sheaf $\bigwedge^d \Omega_{\mathcal{X}/k}$ of differential d-forms. This is an invertible sheaf. If $f : \mathcal{Y} \to \mathcal{X}$ is a proper birational morphism of smooth DM stacks, then there is a natural monomorphism $f^*\omega_\mathcal{X} \to \omega_\mathcal{Y}$. There exists an effective divisor $K_{\mathcal{Y}/\mathcal{X}}$ on $\mathcal{Y}$ with support in the exceptional locus such that $\omega_\mathcal{Y} \cong f^*\omega_\mathcal{X} \otimes \mathcal{O}_\mathcal{Y}(K_{\mathcal{Y}/\mathcal{X}})$. We call $K_{\mathcal{Y}/\mathcal{X}}$ the *relative canonical divisor*. The defining ideal of $K_{\mathcal{Y}/\mathcal{X}}$ is nothing but the Jacobian ideal Jac_f of f.

There exists also a *canonical divisor* $K_\mathcal{X}$, that is, a divisor such that $\mathcal{O}_\mathcal{X}(K_\mathcal{X}) \cong \omega_\mathcal{X}$: Let X be the coarse moduli space of $\mathcal{X}$ endowed with a morphism $f : \mathcal{X} \to X$, $X_0 \subset X$ the smooth locus and $\mathcal{X}_0 \subset \mathcal{X}$ the inverse image of X_0. Since X is normal, $X \setminus X_0$ is of codimension ≥ 2. Let K_{X_0} be a canonical divisor of X_0. Then if we put $K_{\mathcal{X}_0} := f^*K_{X_0} + K_{\mathcal{X}_0/X_0}$, then $K_{\mathcal{X}_0}$ is a canonical divisor of $\mathcal{X}_0$. The unique extension of $K_{\mathcal{X}_0}$ to the whole $\mathcal{X}$ is a canonical divisor of $\mathcal{X}$. Thus a canonical divisor exists. For a morphism of smooth DM stacks $f : \mathcal{Y} \to \mathcal{X}$, we have a equation

$$K_\mathcal{Y} \equiv f^*K_\mathcal{X} + K_{\mathcal{Y}/\mathcal{X}}.$$

If $\mathcal{X}$ is not smooth but only normal and $\mathcal{X}_0 \subset \mathcal{X}$ be the smooth locus, then a canonical divisor $K_\mathcal{X}$ of $\mathcal{X}$ is defined to be a divisor such that $K_\mathcal{X}|_{\mathcal{X}_0}$ is a canonical divisor of $\mathcal{X}_0$.

Theorem 4.2. *Let*

$$\begin{tikzcd} & \mathcal{Y} \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ \mathcal{X} & & \mathcal{X}' \end{tikzcd}$$

be a diagram consisting of smooth DM stacks of pure dimension and tame proper birational morphisms. Let D and D' be $\mathbb{Q}$-divisors on $\mathcal{X}$

and $\mathcal{X}'$ respectively and let W and W' be constructible subsets of $|\mathcal{X}|$ and $|\mathcal{X}'|$ respectively. Then if $f^{-1}(W) = (f')^{-1}(W')$ and

$$f^*D - K_{\mathcal{Y}/\mathcal{X}} = (f')^*D' - K_{\mathcal{Y}/\mathcal{X}'},$$

then

$$\Sigma_W(\mathcal{X}, D) = \Sigma_{W'}(\mathcal{X}', D').$$

Proof. Since

$$\mathfrak{I}_D \circ f_\infty - \operatorname{ord} \operatorname{Jac}_f = \mathfrak{I}_{D'} \circ f'_\infty - \operatorname{ord} \operatorname{Jac}_{f'},$$

the assertion follows from Theorem 3.41. □

Corollary 4.3. *With the notation of Theorem 4.2, if $K_{\mathcal{Y}/\mathcal{X}} = K_{\mathcal{Y}/\mathcal{X}'}$ and $f^{-1}(W) = (f')^{-1}(W')$, then we have*

$$\sum_{\mathcal{V} \subset \mathcal{J}_0 \mathcal{X}} \{|\mathcal{V}|_W\} \mathbb{L}^{\mathrm{sht}(\mathcal{V})} = \sum_{\mathcal{V}' \subset \mathcal{J}_0 \mathcal{X}'} \{|\mathcal{V}'|_{W'}\} \mathbb{L}^{\mathrm{sht}(\mathcal{V}')}.$$

Here the sums run over the connected components and $|\mathcal{V}|_W$ denotes the inverse image of W in $|\mathcal{V}|$.

Proof. It follows from the fact that

$$\Sigma_W(\mathcal{X}, 0) = \sum_{\mathcal{V} \subset \mathcal{J}_0 \mathcal{X}} \{|\mathcal{V}|_W\} \mathbb{L}^{\mathrm{sht}(\mathcal{V})}.$$

□

4.2. **Homological McKay correspondence and discrepancies.** Suppose $k = \mathbb{C}$. Let X be a $\mathbb{Q}$-Gorenstein variety, that is, a normal variety with $\mathbb{Q}$-Cartier canonical divisor K_X. Let $f : Y \to X$ be a resolution of X, that is, Y is smooth and f is proper and birational. Then we can attach a rational number $a(E, X)$ to each exceptional divisor E such that we have a numerical equivalence

$$K_Y \equiv f^*K_X + \sum_{E:\text{exceptional}} a(E, X)E.$$

We call $a(E, X)$ the *discrepancy* of E with respect to X. If $a(E, X)$ is zero for every $E \subset Y$, then Y is said to be a *crepant resolution* of X. The *discrepancy* of X, denoted $\operatorname{discrep}(X)$, is defined as follows:

$$\operatorname{discrep}(X) := \inf\{a(E, X) | Y \to X, E \subset Y \text{ exceptional}\}.$$

Here $Y \to X$ runs over all resolutions of X. It is well-known that either $\operatorname{discrep}(X) = -\infty$ or $\operatorname{discrep}(X) \geq -1$. If $\operatorname{discrep}(X) \geq -1$, then $\operatorname{discrep}(X)$ is equal to the minimum of $a(E, X)$ for exceptional divisors E on a *single* resolution such that the exceptional locus is

simple normal crossing. The discrepancy is an important invariant of singularities in the minimal model program.

Consider a finite subgroup G of $GL_d(\mathbb{C})$ and the quotient variety $X := \mathbb{C}^d/G$. Then X is $\mathbb{Q}$-Gorenstein and has log terminal singularities, that is, $\mathrm{discrep}(X) > -1$. Moreover if $G \subset SL_d(\mathbb{C})$, then X is Gorenstein and has canonical singularities, that is, $\mathrm{discrep}(X) \ge 0$. Let $g \in G$ be an element of order l and let $\zeta_l = \exp(2\pi\sqrt{-1}/l)$. Choosing suitable basis of $\mathbb{C}^d$, we write

$$g = \mathrm{diag}(\zeta_l^{a_1}, \dots, \zeta_l^{a_d}),\ 0 \le a_i \le l-1.$$

Then we define the *age* of g to be

$$\mathrm{age}(g) := \frac{1}{l}\sum_{i=1}^{d} a_i \in \mathbb{Q}.$$

If $g \in \mathrm{SL}_d(\mathbb{C})$, then $\mathrm{age}(g)$ is an integer. Now we can deduce, from Corollary 4.3, the homological McKay correspondence. It was proved by Y. Ito and Reid [IR] for dimension 3 and by Batyrev [Bat] for arbitrary dimension. See [Rei2] for a nice survey of this subject.

Corollary 4.4 (Homological McKay correspondence)**.** *Suppose that* $G \subset \mathrm{SL}_d(\mathbb{C})$ *and there exists a crepant resolution* $Y \to X$*. For an even number* $i \ge 0$*, let* $n_i := \sharp\{g \in \mathrm{Conj}(G) | \mathrm{age}(g) = i/2\}$*. Then*

$$H^i(Y, \mathbb{Q}) \cong \begin{cases} \mathbb{Q}(-i/2)^{\oplus n_i} & (i : \textit{even}) \\ 0 & (i : \textit{odd}). \end{cases}$$

Proof. Let $\mathcal{X} := [\mathbb{C}^d/G]$. Then the natural morphism $\mathcal{X} \to X$ is a proper birational morphism. Furthermore, since $G \subset \mathrm{SL}_d(\mathbb{C})$, $\mathcal{X}$ and X are isomorphic in codimension one. In particular, $K_{\mathcal{X}}$ is the pull-back of K_X. (Hence $\mathcal{X}$ is a crepant resolution in a generalized sense.)

There exists a smooth DM stack $\mathcal{Y}$ and proper birational morphisms $f : \mathcal{Y} \to \mathcal{X}$ and $f' : \mathcal{Y} \to Y$: For example, we can take a resolution of the irreducible component of $Y \times_X \mathcal{X}$ dominating Y and $\mathcal{X}$. For resolutions of DM stacks, see Subsection 4.5. Then we have $K_{\mathcal{Y}/\mathcal{X}} = K_{\mathcal{Y}/Y}$. For $0 \le a_i \le l-1$, if we put

$$a_i' := \begin{cases} l & (a_i = 0) \\ a_i & (a_i \ne 0), \end{cases}$$

then

$$\frac{1}{l}\sum_i (l - a_i') = d - \sharp\{i | a_i = 0\} - \frac{1}{l}\sum_i a_i.$$

From Corollary 4.3,

$$\{Y\} = \sum_{\mathcal{V}\subset I\mathcal{X}} \{\mathcal{V}\}\mathbb{L}^{\mathrm{sht}(\mathcal{V})} = \sum_{g\in \mathrm{Conj}(G)} \{[(\mathbb{C}^d)^g/C_g]\}\mathbb{L}^{d-\dim(\mathbb{C}^d)^g-\mathrm{age}(g)}.$$

Sending these elements to $\hat{K}_0(MHS)$, we have

$$\chi_h(Y) = \sum_{g\in \mathrm{Conj}(G)} \mathbb{L}^{d-\mathrm{age}(g)}.$$

Since $K_0(MHS) \to \hat{K}_0(MHS)$ is injective, the equation holds also in $K_0(MHS)$. Batyrev [Bat] proved that $H^i_c(Y,\mathbb{Q})$ has pure Hodge structure of weight i. Let $m_i := \{g \in \mathrm{Conj}(G)| - d + \mathrm{age}(g) = -i/2\}$. Looking at the weight i part, we have

$$H^i_c(Y,\mathbb{Q}) \cong \begin{cases} 0 & (i : \text{odd}) \\ \mathbb{Q}(-i/2)^{\oplus m_i} & (i : \text{even}). \end{cases}$$

Here we have used the semisimplicity of polarizable pure Hodge structure. The corollary follows from the Poincaré duality. □

Next, for general $G \subset GL_d(G)$ without reflection, we deduce an expression of the discrepancy of $X = \mathbb{C}^d/G$ in terms of ages of $g \in G$.

Corollary 4.5. *Suppose that G contains no reflection. We have an equation*

$$\mathrm{discrep}(X) = \min\{\mathrm{age}(g)|1 \neq g \in G\} - 1.$$

Proof. Let $\mathcal{X} := [\mathbb{C}^d/G]$. and $V \subset |\mathcal{X}|$ the locus of points with nontrivial automorphism group. We take a resolution $f : Y \to X$ which is an isomorphism over the smooth locus of X. Suppose that the exceptional locus $W \subset Y$ is simple normal crossing. From Corollary 4.3, as in the proof of Corollary 4.4, we have

$$\Sigma_W(Y, -K_{Y/X}) = \Sigma_V(\mathcal{X}, 0).$$

The left hand side can be computed explicitly as follows. Write $K_{Y/X} = \sum_{i\in I} e_iE_i$. Note that $e_i > -1$. For $s = (s_i)_{i\in I} \in (\mathbb{Z}_{\geq 0})^I$, let $I_s := \{i \in I|s_i > 0\}$. For any subset $J \subset I$, we define $E^\circ_J := \bigcap_{i\in J} E_i \setminus \bigcup_{i\in I\setminus J} E_i$. Then we have

$$\mu_Y(\bigcap_{i\in I} \mathcal{I}^{-1}_{D_i}(s_i)) = \{E^\circ_{I_s}\}\mathbb{L}^{-\sum s_i}(\mathbb{L} - 1)^{\sharp I_s}.$$

(See [Cra, The proof of Theorem 2.15]. Note that our definition of the motivic measure differs from that of [Cra] by the multiplication of $\mathbb{L}^d$.

See also Remark 3.17.) Hence

$$\Sigma_W(Y, -K_{Y/X}) = \sum_{0 \neq s \in (\mathbb{Z}_{\geq 0})^I} \{E^{\circ}_{I_s}\} \mathbb{L}^{-\sum (e_i+1)s_i} (\mathbb{L}-1)^{\sharp I_s}.$$

Therefore the dimension of the right hand side is equal to

$$\begin{aligned} &\dim(\sum_{i \in I} \{E^{\circ}_{\{i\}}\} \mathbb{L}^{-e_i-1}(\mathbb{L}-1)) \\ &= d - 1 + \max\{-e_i | i \in I\} \\ &= d - 1 - \mathrm{discrep}(X). \end{aligned}$$

On the other hand, the dimension of $\Sigma_V(\mathcal{X}, 0)$ is equal to $d - \min\{\mathrm{age}(g) | 1 \neq g \in G\}$. This proves the assertion. □

4.3. **Orbifold cohomology.** Chen and Ruan [CR] constructed a new kind of cohomology, called the orbifold cohomology, for topological orbifolds. We define the orbifold cohomology for DM stacks, the algebraic counterpart of the topological orbifold.

Definition 4.6. (1) Let $\mathcal{X}$ be a smooth DM stack over $\mathbb{C}$. We define the orbifold cohomology of $\mathcal{X}$ as follows: For each $i \in \mathbb{Q}$,

$$H^i_{orb}(\mathcal{X}, \mathbb{Q}) := \bigoplus_{\mathcal{V} \subset \mathcal{J}_0 \mathcal{X}} H^{i - 2\mathrm{sht}(\mathcal{V})}(\bar{\mathcal{V}}, \mathbb{Q}) \otimes \mathbb{Q}(-\mathrm{sht}(\mathcal{V})).$$

Here by convention, we put $H^i(X, \mathbb{Q}) = 0$ for a variety X and $i \notin \mathbb{Z}$.

(2) Let $\mathcal{X}$ be a smooth DM stack over a finite field k. Let r be the least common multiple of $\mathrm{sht}(\mathcal{V})$, $\mathcal{V} \subset \mathcal{J}_0 \mathcal{X}$. Suppose that $\mathbb{Q}_p(1/r)$ exists. (This holds after replacing k with its finite extension.) We define the p-adic orbifold cohomology of $\mathcal{X} \otimes \bar{k}$ as follows: For each $i \in \mathbb{Q}$,

$$H^i_{orb}(\mathcal{X} \otimes \bar{k}, \mathbb{Q}_p) := \bigoplus_{\mathcal{V} \subset \mathcal{J}_0 \mathcal{X}} H^{i - 2\mathrm{sht}(\mathcal{V})}(\bar{\mathcal{V}} \otimes \bar{k}, \mathbb{Q}_p) \otimes \mathbb{Q}_p(-\mathrm{sht}(\mathcal{V})).$$

Lemma 4.7. *Let $\mathcal{X}$ be a proper smooth DM stack over a perfect field k.*

(1) *If $k = \mathbb{C}$, then $H^i_{orb}(\mathcal{X}, \mathbb{Q})$ is a pure Hodge structure of weight i.*

(2) *If k is a finite field and if for $r \in \mathbb{N}$ as above, $\mathbb{Q}_p(1/r)$ exists, then $H^i_{orb}(\mathcal{X} \otimes \bar{k}, \mathbb{Q}_p)$ is pure of weight i.*

Proof. 1. Each connected component $\mathcal{V}$ of $\mathcal{J}_0 \mathcal{X}$ is a proper smooth DM stack. Then the coarse moduli space $\bar{\mathcal{V}}$ is a proper variety with

quotient singularities. Therefore $H^i(\bar{\mathcal{V}},\mathbb{Q})$ is a pure Hodge structure of weight i and $H^i_{orb}(\mathcal{X},\mathbb{Q})$ is also so.

2. We claim that the constant sheaf $\mathbb{Q}_p$ on $\bar{\mathcal{V}}\otimes\bar{k}$ is pure of weight zero. Since the purity of sheaves is a local property, we may assume $\bar{\mathcal{V}}=M/G$ for some smooth variety M and a finite group G acting on M. Then since $q:M\to M/G$ is finite, $q_*\mathbb{Q}_p$ is pure of weight zero [Del2] and $\mathbb{Q}_p=(q_*\mathbb{Q}_p)^G$ is also so.

Since $\bar{\mathcal{V}}$ is proper, $H^i(\bar{\mathcal{V}}\otimes\bar{k},\mathbb{Q}_p)$ is pure of weight i and $H^i_{orb}(\mathcal{X}\otimes\bar{k},\mathbb{Q}_p)$ is also so. □

The following corollary was conjectured by Ruan [Rua]. A weak version was proved by Lupercio and Poddar [LP] and the author [Yas1] independently.

Corollary 4.8. *Let $\mathcal{X}$ and $\mathcal{X}'$ be proper smooth DM stacks over $k=\mathbb{C}$. Assume that there exist a smooth DM stack $\mathcal{Y}$ and proper birational morphisms $f:\mathcal{Y}\to\mathcal{X}$ and $f':\mathcal{Y}\to\mathcal{X}$ such that $K_{\mathcal{Y}/\mathcal{X}}=K_{\mathcal{Y}/\mathcal{X}'}$. Then we have an isomorphism of Hodge structures*

$$H^i_{orb}(\mathcal{X},\mathbb{Q})\cong H^i_{orb}(\mathcal{X}',\mathbb{Q}),\ \forall i\in\mathbb{Q}.$$

(We do not assert that there exists a natural isomorphism.)

Proof. From Corollary 4.3, we have

$$\sum_{\mathcal{V}\subset\mathcal{J}_0\mathcal{X}}\chi_h(\bar{\mathcal{V}})\mathbb{L}^{\mathrm{sht}(\mathcal{V})}=\sum_{\mathcal{V}'\subset\mathcal{J}_0\mathcal{X}'}\chi_h(\bar{\mathcal{V}}')\mathbb{L}^{\mathrm{sht}(\mathcal{V}')}. \tag{4.1}$$

Define

$$H^{ev,i}_{orb}(\mathcal{X},\mathbb{Q}):=\bigoplus_{\substack{\mathcal{V}\subset\mathcal{J}_0\mathcal{X}\\ i-2\mathrm{sht}(\mathcal{V}):\mathrm{even}}}H^{i-2\mathrm{sht}(\mathcal{V})}(\bar{\mathcal{V}},\mathbb{Q})\otimes\mathbb{Q}(-\mathrm{sht}(\mathcal{V})),$$

and $H^{odd,i}_{orb}(\mathcal{X},\mathbb{Q})$ similarly. Then

$$H^i_{orb}(\mathcal{X},\mathbb{Q})=H^{ev,i}_{orb}(\mathcal{X},\mathbb{Q})\oplus H^{odd,i}_{orb}(\mathcal{X},\mathbb{Q}).$$

Looking at the weight i part of (4.1), we have

$$[H^{ev,i}_{orb}(\mathcal{X},\mathbb{Q})]-[H^{odd,i}_{orb}(\mathcal{X},\mathbb{Q})]=[H^{ev,i}_{orb}(\mathcal{X}',\mathbb{Q})]-[H^{odd,i}_{orb}(\mathcal{X}',\mathbb{Q})].$$

Since the two terms of each hand side do not cancel out, we have $[H^{ev,i}_{orb}(\mathcal{X},\mathbb{Q})]=[H^{ev,i}_{orb}(\mathcal{X}',\mathbb{Q})]$ and $[H^{odd,i}_{orb}(\mathcal{X},\mathbb{Q})]=[H^{odd,i}_{orb}(\mathcal{X}',\mathbb{Q})]$. Hence $[H^i_{orb}(\mathcal{X},\mathbb{Q})]=[H^i_{orb}(\mathcal{X}',\mathbb{Q})]$.

We claim that for an arbitrary proper smooth DM stack $\mathcal{X}$ over $\mathbb{C}$, $H^i_{orb}(\mathcal{X},\mathbb{Q})$ is semisimple. A polarization of $H^{i-\mathrm{sht}(\mathcal{V})}(\bar{\mathcal{V}},\mathbb{Q})$ induces a non-degenerate bilinear form $Q_{\bar{\mathcal{V}}}$ on $H^{i-\mathrm{sht}(\mathcal{V})}(\mathcal{V},\mathbb{Q})\otimes\mathbb{Q}(-\mathrm{sht}(\mathcal{V}))$ for which the $\frac{1}{r}\mathbb{Z}$-indexed Hodge decomposition of $(H^{i-\mathrm{sht}(\mathcal{V})}(\mathcal{V},\mathbb{Q})\otimes\mathbb{Q}(-\mathrm{sht}(\mathcal{V})))$ is orthogonal. We define a bilinear form $Q_{\mathcal{X}}$ on $H^i_{orb}(\mathcal{X},\mathbb{Q})$

to be the direct sum of $Q_{\bar{\mathcal{V}}}$. Then $Q_{\mathcal{X}}$ is non-degenerate and the Hodge decomposition of $H^i_{orb}(\mathcal{X},\mathbb{Q})$ is orthogonal for this. We can see that $H^i_{orb}(\mathcal{X},\mathbb{Q})$ is semisimple like the usual polarizable Hodge structure.

Now the equation above, $[H^i_{orb}(\mathcal{X},\mathbb{Q})] = [H^i_{orb}(\mathcal{X}',\mathbb{Q})]$, implies that $H^i_{orb}(\mathcal{X},\mathbb{Q}) \cong H^i_{orb}(\mathcal{X}',\mathbb{Q})$. □

Corollary 4.9. *Assume that k is a finite field. Let $\mathcal{X}$ and $\mathcal{X}'$ be proper smooth DM stacks whose p-adic orbifold cohomology groups can be defined. Assume that there exist a smooth DM stack $\mathcal{Y}$ and tame proper birational morphisms $f : \mathcal{Y} \to \mathcal{X}$ and $f' : \mathcal{Y} \to \mathcal{X}$ such that $K_{\mathcal{Y}/\mathcal{X}} = K_{\mathcal{Y}/\mathcal{X}'}$. Then we have the following isomorphisms of Galois representations:*

$$H^i_{orb}(\mathcal{X}\otimes_k \bar{k},\mathbb{Q}_p)^{ss} \cong H^i_{orb}(\mathcal{X}'\otimes_k \bar{k},\mathbb{Q}_p)^{ss},\ \forall i \in \mathbb{Q}.$$

Proof. Similar as Corollary 4.8 except for the semisimplicity. □

4.4. **Convergence and normal crossing divisors.**

Definition 4.10. Let $\mathcal{X}$ be a smooth DM stack and $D = \sum_{i=1}^n D_i$ a divisor of $\mathcal{X}$ with D_i distinct prime divisors. We say that D is *normal crossing* if the pull-back of D to an atlas of $\mathcal{X}$ is (analytically) normal crossing.

Let $\mathcal{X}$ be a smooth DM stack and $x \in \mathcal{X}(\bar{k})$. Let $\hat{\mathcal{X}} = [\operatorname{Spec} \bar{k}[[x_1,\dots,x_d]]/G]$ be the completion of $\mathcal{X}$ at x. If D is a normal crossing divisor on $\mathcal{X}$, then for suitable local coordinates $x_1,\dots,x_d$, the pull-back of D to $\hat{\mathcal{X}}$ is defined by a monomial $x_1x_2\cdots x_c$, $c \le d$.

Definition 4.11. A normal crossing divisor D is said to be *stable normal crossing* if for every $x \in \mathcal{X}(\bar{k})$, every irreducible component of its pull-back to $\operatorname{Spec}\bar{k}[[x_1,\dots,x_d]]$ is stable under the G-action.

If D is stable normal crossing and $l \in \mathbb{N}$ is prime to the characteristic of k, then for each embedding $a : \boldsymbol{\mu}_l \hookrightarrow G$, we can choose local coordinates $x_1,\dots,x_d$ so that the $\boldsymbol{\mu}_l$-action is linear and diagonal and th pull-back of D is defined by a monomial simultaneously: Suppose that the pull-back of D is defined by $x_1\cdots x_c$. Since each irreducible component is stable under the $\boldsymbol{\mu}_l$-action, for $1 \le i \le c$ and $\zeta \in \boldsymbol{\mu}_l$, $\zeta(y_i)$ lies in the ideal (y_i). Let $\bar{\zeta}$ be the linear part of ζ, namely

$$\zeta(y_i) = \bar{\zeta}(y_i) + (\text{terms of order} \ge 2).$$

If we replace y_i with

$$y_i' := \sum_{\zeta\in\boldsymbol{\mu}_l} \bar{\zeta}^{-1}\zeta(y_i),$$

then the $\boldsymbol{\mu}_l$-action is linear. We have the identity of ideals $(y_i) = (y'_i)$, $1 \le i \le c$, hence the pull-back of D is still defined by a monomial. Then the $\boldsymbol{\mu}_l$-action must be diagonal about coordinates $x_1, \dots, x_c$. Replacing the rest coordinates, we can diagonalize the action.

Remark 4.12. The stable normal crossing divisor is a notion for stacks corresponding the G-normal pair in [Bat].

Proposition 4.13. *Let $\mathcal{X}$ be a smooth DM stack, $D = \sum_{i=1}^m u_i D_i$ a $\mathbb{Q}$-divisor with stable normal crossing support and $W \subset |\mathcal{X}|$ a constructible subset. Then $\Sigma_W(\mathcal{X}, D) \ne \infty$ if and only if $u_i < 1$ for every i with $D_i \cap W \ne \emptyset$.*

Proof. From the semicontinuity of dimension of fibers, it suffices to show the proposition in the case where $W = \{x\}$ with $x \in \mathcal{X}(\bar{k})$. Shrinking $\mathcal{X}$, we can assume that every D_i contains x. Take the completion

$$\hat{\mathcal{X}} := [\operatorname{Spec} \bar{k}[[x_1, \dots, x_d]]/G]$$

of $\mathcal{X}$ at x. Take an embedding $a : \boldsymbol{\mu}_l \hookrightarrow G$ and choose local coordinates $x_1, \dots, x_d$ so that $\boldsymbol{\mu}_l$ acts linearly and diagonally and the pull-back of D_i is defined by $x_i = 0$ for $1 \le i \le c \le d$ and $x \notin D_i$ for $i > c$. Suppose that $\zeta \in \boldsymbol{\mu}_l$ acts by $\operatorname{diag}(\zeta^{a_1}, \dots, \zeta^{a_d})$, $1 \le a_i \le l$.

Let v be the $\bar{k}$-point of $\mathcal{J}_0\mathcal{X}$ corresponding (x, a). Then for $0 \le n \le \infty$, the fiber $(\mathcal{J}_n\mathcal{X})_v$ of $\mathcal{J}_n\mathcal{X} \to \mathcal{J}_0\mathcal{X}$ over v is identified with

$$\operatorname{Hom}^{\boldsymbol{\mu}_l}_{\bar{k}[[t]]}(\bar{k}[[x_1, \dots, x_d]], \bar{k}[[t]]/t^{nl+1}).$$

If $\sigma \in (\mathcal{J}_\infty\mathcal{X})_v$ and $\sigma(y_i) = \Sigma_{j \ge 0} \sigma_{ij} t^{lj+a_i}$, then the order of the ideal (x_i) along the twisted arc σ is

$$\frac{a_i}{l} + \min\{j | \sigma_{ij} \ne 0\}.$$

For a multi-index $s = (s_1, \dots s_c) \in (\mathbb{Z}_{\ge 0})^c$, we define $V_s \subset (\mathcal{J}_\infty\mathcal{X})_v$ to be the set of σ such that $\min\{j | \sigma_{ij} \ne 0\} = s_i$ for every $1 \le i \le c$. Then

$$\int_{\pi_0^{-1}(\bar{v})} \mathbb{L}^{\mathfrak{I}_D + \mathfrak{s}_\mathcal{X}} d\mu_\mathcal{X} = \sum_s \mu_\mathcal{Y}(\bar{V}_s) \mathbb{L}^{\sum_{i=1}^c u_i(s_i + a_i/l)}. \tag{4.2}$$

Here $\bar{v} \in |\mathcal{J}_0\mathcal{X}|$ is the image of v and $\bar{V}_s \subset |\mathcal{J}_\infty\mathcal{X}|$ is the image of V_s. For sufficiently large n,

$$\dim \pi_n(\bar{V}_s) = (d-c)n + \sum_{i=1}^c (n - s_i) = dn - \sum_{i=1}^c s_i.$$

Hence

$$\dim \mu_\mathcal{Y}(\bar{V}_s) \mathbb{L}^{\sum_{i=1}^c u_i(s_i + a_i/l)} = (u_i - 1)s_i + C.$$

Here C is a constant independent of s. Hence the right hand side of (4.2) converges if and only if $u_i < 1$ for every i. $\square$

Proposition 4.14. *Let $\mathcal{X}$ be a smooth DM stack and D a normal crossing divisor of $\mathcal{X}$. Then there exist a smooth DM stack $\mathcal{Y}$ and a representable proper birational morphism $f : \mathcal{Y} \to \mathcal{X}$ such that f is an isomorphism over $\mathcal{X} \setminus D$ and $f^{-1}(D)$ is stable normal crossing.*

Proof. Let Δ be an irreducible component of D. Note that Δ can be singular. For each $\bar{k}$-point $p \in \Delta$, we define the *index* $i_\Delta(p)$ as follows: Let $[\operatorname{Spec} \bar{k}[[x_1, \dots, x_d]]/G]$ be the completion of $\mathcal{X}$ at p. Choose coordinates so that the pull-back of Δ is defined by $x_1 \cdots x_c$. Then $i_\Delta(p) := c$.

The function

$$i_\Delta : \Delta(\bar{k}) \to \mathbb{N}$$

is upper semi-continuous. The locus $V \subset \Delta$ of the points of the maximum index is a closed smooth substack defined over k. Let $\mathcal{X}' \to \mathcal{X}$ be the blow-up along V, E the exceptional divisor, D' the strict transform of D and Δ' the strict transform of Δ. Then $D' \cup E$ is normal crossing. For every irreducible component E_1 of E and for every $p \in E_1(\bar{k})$, we have $i_{E_1}(p) = 1$. Moreover the maximum value of the function $i_{\Delta'}$ is less than that of i_Δ. Hence repeating blow-ups, we obtain a proper birational morphism $f : \mathcal{Y} \to \mathcal{X}$ such that every $\bar{k}$-point of $f^{-1}(D)$ has index one with respect to every irreducible component of $f^{-1}(D)$. It means that $f^{-1}(D)$ is stable normal crossing. $\square$

Proposition 4.15. *Let $\mathcal{X}$ be a smooth DM stack, $D = \sum u_i D_i$ a $\mathbb{Q}$-divisor with (not necessarily stable) normal crossing support and $W \subset |\mathcal{X}|$ a constructible subset. Then $\Sigma_W(\mathcal{X}, D) \neq \infty$ if and only if $u_i < 1$ for every i with $D_i \cap W \neq \emptyset$.*

Proof. Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism as in Proposition 4.14. Then by Theorem 4.2, we have

$$\Sigma_W(\mathcal{X}, D) = \Sigma_{f^{-1}(W)}(\mathcal{Y}, f^*D - K_{\mathcal{Y}/\mathcal{X}}).$$

Shrinking $\mathcal{X}$, we may assume that every D_i meets W. By the standard calculation in the minimal model program, if $u_i < 1$ for every i, then coefficients in $f^*D - K_{\mathcal{Y}/\mathcal{X}}$ are also all < 1. Therefore from Proposition 4.13, the $\Sigma_W(\mathcal{X}, D) = \Sigma_{f^{-1}(W)}(\mathcal{Y}, f^*D - K_{\mathcal{Y}/\mathcal{X}}) \neq \infty$. If $u_i \geq 1$ for some i, the coefficient of the strict transform of D_i in $f^*D - K_{\mathcal{Y}/\mathcal{X}}$ is also $u_i \geq 1$. Hence again from Proposition 4.13, $\Sigma_W(\mathcal{X}, D) = \infty$. $\square$

4.5. **Generalization to singular stacks.** From now on, we assume that the base field k is of characteristic zero.

Thanks to Hironaka [Hir], for every reduced variety X, there exists a resolution of singularities, that is, a proper birational morphism $Y \to X$ with Y smooth. Villamayor [Vil1] [Vil2] constructed a resolution algorithm commuting with smooth morphisms. See also [BM], [EV]. Let $\mathcal{X}$ be a reduced DM stack and M an atlas. Then we obtain a groupoid space $N := M \times_{\mathcal{X}} M \rightrightarrows M$. The associated stack of this groupoid space is canonically isomorphic to $\mathcal{X}$. Let $\tilde{N}$ and $\tilde{M}$ be smooth varieties obtained from N and M respectively by a resolution algorithm commuting with étale morphisms. Then we obtain a groupoid space $\tilde{N} \rightrightarrows \tilde{M}$. Its associated DM stack $\tilde{\mathcal{X}}$ is smooth and the natural morphism $\tilde{\mathcal{X}} \to \mathcal{X}$ is representable, proper and birational. Thus for every reduced DM stack $\mathcal{X}$, there exists a representable proper birational morphism $\mathcal{Y} \to \mathcal{X}$ with $\mathcal{Y}$ smooth.

Let $\mathcal{X}$ be a normal DM stack and D a $\mathbb{Q}$-divisor on $\mathcal{X}$. Suppose that $K_{\mathcal{X}} + D$ is $\mathbb{Q}$-Cartier. Then we say that the pair $(\mathcal{X}, D)$ is a *log DM stack*. For a resolution $f : \mathcal{Y} \to \mathcal{X}$, we define a $\mathbb{Q}$-divisor E on $\mathcal{Y}$ by

$$K_{\mathcal{Y}} + E \equiv f^*(K_{\mathcal{X}} + D).$$

Definition 4.16. Let the notations be as above. Let $W \subset |\mathcal{X}|$ be a constructible subset. Then we define an invariant

$$\Sigma_W(\mathcal{X}, D) := \Sigma_{f^{-1}(W)}(\mathcal{Y}, E).$$

This is independent of the choice of resolutions, thanks to Theorem 4.2.

Definition 4.17. We say that the pair $(\mathcal{X}, D)$ is *Kawamata log terminal (KLT for short)* if for every representable resolution $\mathcal{Y} \to \mathcal{X}$, every coefficient of the divisor E defined as above is less than one.

In fact, we can see if $(\mathcal{X}, D)$ is KLT by looking at only one resolution with E normal crossing. The pair $(\mathcal{X}, D)$ is KLT if and only if for an atlas $M \to \mathcal{X}$ and the pull-back D' of D to M, the pair (M, D') is KLT.

Proposition 4.18. *The invariant $\Sigma_W(\mathcal{X}, D)$ is not infinite if and only if $(\mathcal{X}, D)$ is KLT around W (that is, for some open substack $\mathcal{X}_0 \subset \mathcal{X}$ containing W, $(\mathcal{X}_0, D|_{\mathcal{X}_0})$ is KLT).*

Proof. It is a direct consequence of Proposition 4.15 and the definition of the invariant. ☐

Theorem 4.19. *Let $(\mathcal{X}, D)$ and $(\mathcal{X}', D')$ be log DM stacks. Let W and W' be constructible subsets of $|\mathcal{X}|$ and $|\mathcal{X}'|$ respectively. Assume*

that there exist a smooth DM stack $\mathcal{Y}$ and proper birational morphisms $f : \mathcal{Y} \to \mathcal{X}$ and $f' : \mathcal{Y} \to \mathcal{X}$ such that $f^*(K_{\mathcal{X}} + D) \equiv f^*(K_{\mathcal{X}'} + D')$ and $f^{-1}(W) = (f')^{-1}(W')$. Then we have

$$\Sigma_W(\mathcal{X}, D) = \Sigma_{W'}(\mathcal{X}', D').$$

Proof. It follows from Theorem 4.2. □

4.6. **Invariants for varieties.** If X is a variety, then we can describe the invariant $\Sigma_W(X, D)$ with the motivic integration over X itself. It gives us a canonical expression of the invariant.

Let X be a normal variety and D a $\mathbb{Q}$-divisor on X such that $m(K_X + D)$ is Cartier and mD is an integral divisor for some $m \in \mathbb{N}$. Let $\mathcal{K}$ be the sheaf of total quotient rings of $\mathcal{O}_X$. There exists a fractional ideal sheaf $\mathcal{G} \subset \mathcal{K}$ such that $\mathcal{G} = \mathcal{O}_X(-mD)$ outside the singular locus of X, and the canonical isomorphism $(\Omega_X^d)^{\otimes m} \cong \mathcal{G}\mathcal{O}_X(m(K_X + D))$ outside the singular locus of X extends to an epimorphism $(\Omega_X^d)^{\otimes m} \twoheadrightarrow \mathcal{G} \cdot \mathcal{O}_X(m(K_X + D))$ all over X.

Definition 4.20. Let $\mathcal{I} \subset \mathcal{K}$ be a fractional ideal sheaf. We define the *order function* of $\mathcal{I}$ as follows;

$$\operatorname{ord} \mathcal{I} : J_\infty X \to \mathbb{Z} \cup \{\infty\}$$
$$(\gamma : \operatorname{Spec} K[[t]] \to X) \mapsto \begin{cases} n & (\gamma^{-1}\mathcal{I} = (t^n) \subset K((t))) \\ \infty & (\gamma^{-1}\mathcal{I} = (0)). \end{cases}$$

If A is a Cartier divisor and $\mathcal{I}_A = \mathcal{O}_{\mathcal{X}}(-A)$ is the corresponding fractional ideal sheaf, then we have $\operatorname{ord} \mathcal{I}_A = \mathfrak{I}_A$, at least outside a negligible subset.

We have the following expression of the invariant $\Sigma_W(X, D)$.

Proposition 4.21. *Let $W \subset X$ be a constructible subset. We have*

$$\Sigma_W(X, D) = \int_{\pi_0^{-1}(W)} \mathbb{L}^{(1/m)\operatorname{ord} \mathcal{G}} d\mu_X.$$

Proof. Let $f : Y \to X$ be a resolution of singularities. We define a $\mathbb{Q}$-divisor E on Y by

$$K_Y + E \equiv f^*(K_X + D).$$

Let $\mathcal{I} = \mathcal{O}_X(-mE)$. We have natural morphisms

$$f^*\mathcal{O}_X(m(K_X + D)) \cong \mathcal{I}^{-1}\omega_Y^{\otimes m}$$
$$f^*\Omega_X^d \twoheadrightarrow \operatorname{Jac}_f \omega_Y$$
$$(\Omega_X^d)^{\otimes m} \twoheadrightarrow \mathcal{G}\mathcal{O}_X(m(K_X + D)).$$

Therefore we have

$$(\mathrm{Jac}_f)^m = f^{-1}\mathcal{G} \cdot \mathcal{I}^{-1}.$$

We obtain an equation of measurable functions

$$\frac{1}{m}\mathrm{ord}\,\mathcal{G} \circ f_\infty - \mathrm{ord}\,\mathrm{Jac}_f = \mathfrak{I}_E.$$

Now the proposition follows from Theorem 3.33. □

Remark 4.22. This kind of integration over a singular variety was considered in [DL2] (see also [Loo]) in relation to the McKay correspondence and in [Yas2] and [EMY] in relation to the discrepancy of singularities.

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